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Extensions of the classical fighter-bomber duel

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Extensions of the classical fighter-bomber duel

by

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CHAPTER 1. INTRODUCTION

This research is directed toward extensions of the classical fighter-bomber duel, i.e., a silent duel between a fighter capable of firing a single missile and a bomber capable of maintaining continuous fire.

Duels have traditionally been categorized according to whether they are "silent" or "noisy." Noisy duels are called so, because the opponents are assumed to be instantaneously aware of the opponent's firing; as such, they are duels of high, if not "perfect" information. Silent duels are called so, because the opponents are assumed not to know (assuming, they survive) if or when the opponent has fired.

The study of the noisy duel mostly has dealt with the case of a duel between agents each equipped with a finite number of bullets. As such, it has served a number of purposes:

(1) The discretized noisy duel has served as a vehicle for illustrating the algorithmic method applicable to multi-stage games, as a means for obtaining a behavioral saddle point of a game of perfect recall [31], [38].

(2) The discretized noisy duel and its asymptotic form has provided a stochastic representation of the "split-second anticipation" phenomenon [11] that marks the continuous versions of such duels.

(3) The continuous version of the noisy duel has served to illustrate the concept of

"ϵ – good strategy" [7], [14], [15].

The silent duel to be studied here leaves behind matters of informational structure – be it the "noise" that keeps the players of the discretized noisy duel continually aware of the opponent's past actions, or the supreme act of information gathering, i.e., split-second anticipation, required of the weaker player in the continuous noisy duel. But complexity of information structure and ease of algorithmic solution is now traded for a fairly complex solution in the context of an information structure involving no information gathering at all.

The contribution of this research consists of the analysis of saddle point coordinate strategies for the Weiss-Gillman model (Chapter 3) and the Karlin model (Chapter 4) , a late duel start model with special structure (Chapter 5), a multiple missile model (Chapter 6), alternative payoff functions (Chapter 7), a non-zero sum version (Chapter 8), and a cooperative version (Chapter 9).

Prior work in this area includes the work of Weiss [45] who first examined the fighter-bomber duel, in the course of munition studies conducted at the Aberdeen Proving Ground. The work of Bellman and Blackwell [5] , and Blackwell and Shiffman [6] , and Gillman [17] also is related to the optimal strategies for the fighter and bomber. The interpretation of this problem as an advertising competition was considered by Gillman [18] . Karlin [23] also has addressed this problem in advertising competition terms, but with a slightly different formulation. Further researches on duels have been done by many researchers (see, for example, [8], [29], [39], and [40]).

The organization of this dissertation is as follows:

In Chapter 2 the classical fighter-bomber duel is described, including the relevant notation and assumptions, with the intent of extending this duel in the following chapters. The classical case is that in which the fighter has one missile. The bomber has much small-caliber ammunition for protecting itself from the fighter.

Chapter 3 deals with the Weiss-Gillman formulation of the classical fighter-bomber duel.

Chapter 4 deals with the Karlin formulation of the classical fighter-bomber duel.

Chapter 5 studies the relation between two duel solutions corresponding respectively to two related fighter "lethality functions."

Chapter 6 deals with the duel in which the fighter possesses several identical missiles.

Chapter 7 discusses the duel with certain alternative payoff functions such as a "fighter-perspective" payoff and "hybrid" payoff function. The latter deals with a payoff function that mixes bomber-kill and fighter-survival probability.

Chapter 8 deals with a non-zero sum version of the fighter-bomber duel. We find the attractive feature that equilibrium points and maximin points coincide.

Chapter 9 deals with a cooperative bargaining view of the fighter-bomber duel, which may be especially relevant to Gillman's advertising competition formulation.

Chapter 10 presents conclusions and recommendations for further research in related topics.

CHAPTER 2. CLASSICAL FIGHTER-BOMBER DUEL

The Weiss–Gillman Model

Consider a situation where a fighter, Player I, is attacking a bomber, Player II. The fighter is armed with one missile, possessing lethality function $F(r)$ decreasing on $[0, R]$, presumably from near 1 to near 0. The bomber is armed with a great weight (say, A ozs.) of armament, with lethality function $p(r)$, also decreasing to near zero on $[0, R]$. $p(r)$ is to be thought of in these terms: When da ounces of ammunition are expended by the bomber at range r, there results the probability $p(r)da$ that the fighter is killed.

A strategy for the fighter is a cumulative distribution function $\sigma(r)$ on $[c, R]$ giving the probability distribution from which the fighter initially selects its firing range. A strategy for the bomber is the ammunition distribution density $\tau(r)$ according to which the bomber plans to distribute the ammunition store A over $[c, R]$, with $\tau(r)dr = da$, the number of ounces da allotted to the range interval dr, satisfying $\int_c^R \tau(r)\,dr \leq A$, where the minimum closing range c is not necessarily zero.

The objective function $M(\sigma, \tau)$ is taken to be the probability that the bomber is killed, when the fighter fires its missile with probability distribution $\sigma(r)$, and the bomber uses firing intensity function $\tau(r)$, computed as

$$M(\sigma,\tau) = \int_c^R F(r)e^{-\int_r^R \tau(s)p(s)\,ds}\,d\sigma(r),$$

which is not an unreasonable assessment, since $\int_r^R \tau(s)p(s)\,ds$ is the expected number of potential kills gotten off by the bomber up to range r. We initially make the assumption that there is a range $r_O \leq R$ such that

$$\int_c^{r_O} \frac{-F'(s)}{F(s)p(s)}\,ds = A. \tag{2.1}$$

In essence, Equation 2.1 guarantees that a certain natural bomber minimax strategy given below calls for rapid enough expenditure of ammunition to insure that the amount of ammunition A is expended on $[c, R]$.

The notation is summarized as follows:

R: Initial range for the duel.

$F(r)$: The probability that fighter's missile, if fired at range r, is lethal to the bomber. (positive and non-increasing in r)

$p(r)$: A function, positive and non-increasing in r, such that, when da ounces of ammunition are expended by the bomber at range r, there results the probability $p(r)da$ that the fighter is killed.

$\sigma(r)$: Strategy for the fighter (probability that the fighter fires at a range less than or equal to r).

$\tau(r)$: Strategy for the bomber (instantaneous firing intensity, in units of ozs./ft., such that, when the bomber employs $\tau(r)$, there results a probability $\tau(r)p(r)dr$ that the fighter is killed between r and $r + dr$).

c: Minimum closing range for the fighter and bomber.

$\int_r^R \tau(s)p(s)\,ds$: Expected number of lethal shots gotten off by the bomber in range $[r, R]$.

$e^{-\int_r^R \tau(s)p(s)\,ds}$: Probability that the fighter survives up to range r.

The assumptions are as follows:

a. $R \geq r_o$.

b. The fighter has a single missile available.

c. The bomber can approximate continuous fire and has "A" amount of ammunition, of which the bomber may spend all or some.

d. The fighter does not know whether or not, or how much, the bomber has fired or is firing.

e. There is no cumulation of damage, and statistical independence of bomber shots.

f. Kill probability does not decrease with closing range.

g. Weapon velocity is infinite.

h. The only premium of the duel is bomber kill.

i. Both parties are aware of the duel parameters $p(r)$, $F(r)$, R, and A.

Weiss [45] and Gillman [18] examined the model which we described above. A saddle point for the duel is as follows:

The optimal strategy for the fighter is

$$\sigma^o(r) : \begin{cases} \dfrac{p(r_o)}{p(r)} & \text{for } c \leq r < r_o, \\ 1 & \text{for } r_o \leq r \leq R. \end{cases}$$

The optimal strategy for the bomber is

$$\tau^o(r) : \begin{cases} \dfrac{-F'(r)}{F(r)p(r)} & \text{for } c \leq r < r_o, \\ 0 & \text{for } r_o \leq r \leq R. \end{cases}$$

If both opponents adopt the optimal strategies, the probability of bomber destruction is $F(r_o)$: That is,

$$M(\sigma^o, \tau^o) = F(r_o).$$

The Karlin Model

Karlin [23] also has addressed essentially this problem as an advertising battle. Consider a competition between two businessmen, Mr. Big (bomber) and Mr. Little (fighter). Mr. Big is prosperous and has a stable supply of customers; Mr. Little is on the verge of bankruptcy. Now at time 0 a potential customer arrives on the scene. He may decide not to buy at all, but if he does buy he will place a large order with only one of the concerns. The order is sufficiently large to put Mr. Little back on his feet, but to remain in business Mr. Little must obtain the order by a certain time - call it time 1. Mr. Little wins if he secures the customer within the allotted time; Mr. Big wins if the customer does not buy or, a fortiori, if Mr. Big receives the order. By proper interpretation of the utility objectives of the participants the game is zero–sum [23].

The assumptions for this model are as following:

(1) Only Mr. Big or Mr. Little can win the order, and "sales pitch" will be the decisive factor in determining the customer's decision.

(2) The customer's resistance to buying decreases with time.

(3) The customer's psychology is such that only the current sales effort exerts any influence on him.

(4) Mr. Big is able to apply continuous sales pressure, while Mr. Little can mount only a single attempt.

We formulate the campaign in a mathematically continuous manner. Mr. Little selects a time t in $[0,1]$ at which to try to convince the customer, while Mr. Big selects a rate $\tau(t)$ of badgering the customer which is subject to the restrictions

$$0 \leq \tau(t) \leq 1,$$

$$\int_0^1 \tau(t)\, dt = \delta < 1,$$

where the first of these two conditions differentiates Karlin's formulation from that of Weiss–Gillman and where δ is related to the money Mr. Big has allotted to his campaign; i.e., if Mr. Big is spending at rate $\tau(t)$, then the money will last for the time interval 1 [23].

There are "customer susceptibility" functions $F(t)$ and $p(t)$, respectively for Mr. Little and Mr. Big, that are non-decreasing in time. Specifically, $F(t)$ is the probability that Mr. Little will make the sale at time t if he chooses to approach the customer at that time; on the other hand $p(t)$ is such that an expenditure of da by Mr. Big at time t, results in a probability of $p(t)da$ that the customer will succumb to Mr. Big. The times 0 and 1 are defined as starting and ending time for the sales competition, respectively.

The objective function (as seen by Mr. Little) is as follows, with randomized strategy $\sigma(t)$ for Mr. Little:

$$M(\sigma,\tau) = \int_0^1 F(t) e^{-\int_0^t \tau(s)p(s)\, ds}\, d\sigma(t).$$

The solution given below applies only when $\frac{F'(t)}{F(t)p(t)}$ is strictly decreasing in time t [23].

Define the following quantities and function:

$$
\begin{aligned}
w(t) &= \min\{1, \frac{F'(t)}{F(t)p(t)}\}. \\
d &= \min\{x : \frac{F'(x)}{F(x)p(x)} \geq 1\}. \\
\delta_O &= \int_0^1 w(t)\,dt. \\
\delta &= \int_e^1 w(t)\,dt.
\end{aligned}
$$

The solution of the sales competition now is described by Karlin [23], as follows, in terms of the quantities $w(t)$, e, and d. He considers three cases:

When $\delta_O > \delta$, $e < d$,

the optimal strategy for Mr. Little (fighter) is

$$
\sigma^O(t) : \begin{cases} 0 & \text{for } 0 \leq t < d, \\ 1 - \dfrac{p(e)}{p(t)} & \text{for } d \leq t < 1, \\ 1 & \text{for } t = 1; \end{cases}
$$

the optimal strategy for Mr. Big (bomber) is

$$
\tau^O(t) : \begin{cases} 0 & \text{for } 0 \leq t < e, \\ w(t) & \text{for } e \leq t \leq 1. \end{cases}
$$

When $\delta_O > \delta$, $d < e$,

the optimal strategy for Mr. Little (fighter) is

$$
\sigma^O(t) : \begin{cases} 0 & \text{for } 0 \leq t < e, \\ 1 - \dfrac{p(e)}{p(t)} & \text{for } e \leq t < 1, \\ 1 & \text{for } t = 1; \end{cases}
$$

the optimal strategy for Mr. Big (bomber) is

$$\tau^{o}(t) : \begin{cases} 0 & \text{for } 0 \leq t < e, \\ w(t) & \text{for } e \leq t \leq 1. \end{cases}$$

When $\delta_o = \delta$, $e = 0$,

the optimal strategy for Mr. Little (fighter) is a degenerate distribution concentrating at $t = d$:

the optimal strategy for Mr. Big (bomber) is any $\tau^{o}(t)$ such that

$$\begin{aligned} \tau^{o}(t) &= 1 \qquad (0 \leq t \leq d), \\ \frac{F'(t)}{F(t)p(t)} \leq \tau^{o}(t) &\leq 1 \qquad (d \leq t \leq 1), \\ \int_0^1 \tau^{o}(t)\, dt &= \delta. \end{aligned}$$

If $\dfrac{F'(t)}{F(t)p(t)}$ is not monotonic, the optimal strategy for Mr. Big consists of intervals in which $\tau^{o}(r) = w(t)$ alternating with intervals in which $\tau^{o}(t) = 0$.

CHAPTER 3. ANALYSIS OF THE WEISS-GILLMAN MODEL

Introductory Remarks

This chapter deals with the optimal strategies suggested by Weiss–Gillman for the fighter and bomber. We re-derive these solutions, using the so-called constancy-positivity principle, as well as the Lagrangian saddle point approach to constrained optimization. Both the constancy-positivity principle and Lagrangian saddle point approach are in their infinite-dimensional versions. In particular, we shall first see how the bomber's optimal strategy is suggested by the constancy-positivity principle, and then a sense in which the optimization problem involved in establishing a saddle point solution touches on Lagrangian optimization.

In this chapter we assume that the fighter and bomber open fire at the same range R which is at least as large as the range r_O, in accordance with the previous chapter, and the quantity $p(R) > 0$, and $F(R) > 0$. The first two section below establish candidate saddle point coordinate strategies σ^O and τ^O, under the assumption that τ^O is derivable by the constancy–positivity principle. The third section below then verifies that the candidate strategies σ^O and τ^O do indeed constitute a saddle point for the game in which the duel starts at range R. The fourth section suggests a saddle point coordinate strategy for the fighter with altered restriction on the bomber. The last section shows saddle point coordinate strategies for a late duel start model with

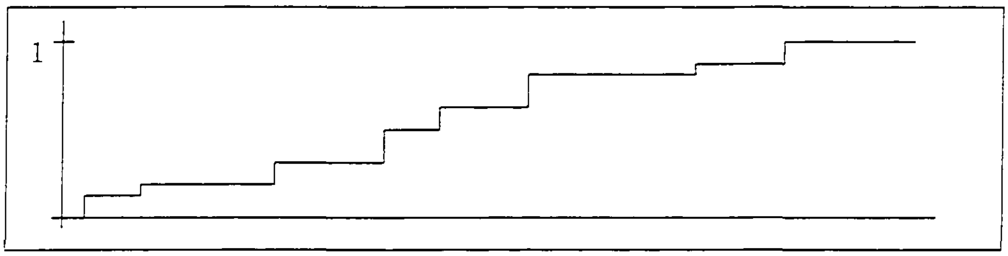

Figure 3.1: Step Function

initial burst possibility for the bomber.

We will need to consider strategies for the fighter that are cumulative distribution functions possessing both discrete and absolutely continuous parts.

Purely discrete cumulative distribution functions are step functions which are shown in Figure 3.1 indicating that the random variable being described takes on only a "countable" set of values, and each of them with a certain specified probability. Examples are the Poisson or the binomial cumulative distribution function. Average or expectation of $\sin X$, X Poisson, is given by

$$\sum_{i=0}^{\infty}(\sin(i))\frac{\lambda^i e^{-\lambda}}{i!}.$$

Purely absolutely continuous cumulative distribution functions are smooth functions which are shown in Figure 3.2 indicating that the random variable being described is capable of taking on a continuum of values, with any interval assigned probability given by the definite integral of a density function over that interval. An example is the normal cumulative distribution function, with the familiar density $\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$. Averages or expectations with respect to such cumulative distribution functions are

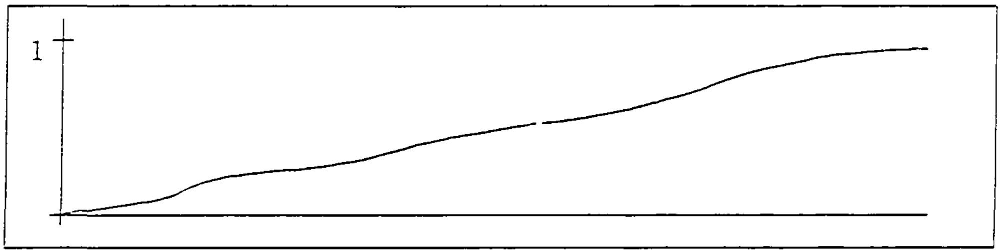

Figure 3.2: Absolute-Continuous Distribution Function

expressed as integrals involving the density. Thus the expectation of sin X, X normal, is given by

$$\int_{-\infty}^{+\infty} [\sin(t)] \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} \, dt.$$

We shall need to deal with cumulative distribution functions σ that are partly discrete and partly absolutely continuous. Such cumulative distribution functions may be thought of in at least these two ways:

As a cumulative distribution function with both steps and smooth portions which can be shown in Figure 3.3. Or, in more explicit fashion, as a cumulative distribution function equal to a weighted average

$$\sigma(t) = \theta\sigma_d(t) + (1 - \theta)\sigma_{ac}(t)$$

of a discrete cumulative distribution function $\sigma_d(t)$ which can be seen in Figure 3.4, and an absolutely continuous cumulative distribution function $\sigma_{ac}(t)$ which can be described in Figure 3.5, with density function, namely, $\bar{g}(t)$.

Averages or expectations with respect to such cumulative distribution functions are expressed as integrals with density $(1 - \theta)\bar{g}(t) \equiv g(t)$, plus summations. Thus

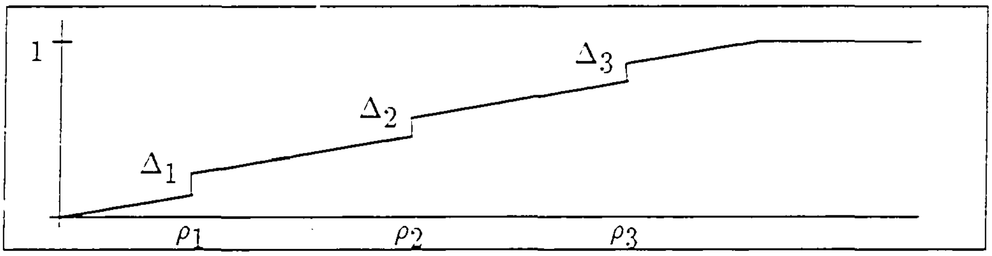

Figure 3.3: CDF with Steps and Smooth Portions

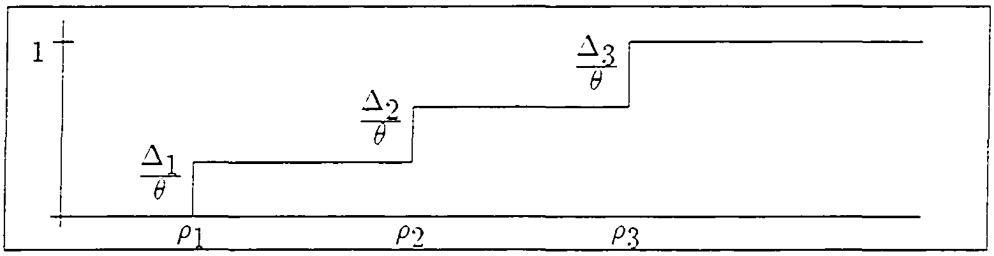

Figure 3.4: Discrete Distribution Function

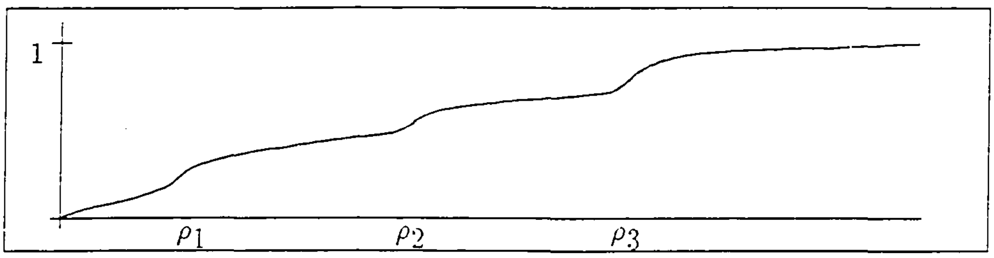

Figure 3.5: Continuous Distribution Function

the expectation of sin X, X distributed as in Figure 3.3, is given by

$$\int_{-\infty}^{-\infty} (\sin(t))(g(t))\,dt - \sum_{i=1}^{3} (\sin(\rho_i))(\Delta_i).$$

Such expectations commonly are given the Stieltjes integral designation, say

$$\int_{-\infty}^{+\infty} \sin(t)\,d\sigma(t).$$

We also note that , when $\sigma(t) = 0$ for $t < 0$, then

$$\sigma(r) = \int_{-\infty}^{r} d\sigma(t) = \int_{0}^{r} d\sigma(t) \text{ for any } r \geq 0.$$

Candidate Optimal Bomber Strategy by Constancy-Positivity Principle

In this section we will see how the bomber's optimal strategy ($\tau^{o}(r)$) is suggested by the constancy-positivity principle.

We begin by recalling what is meant by a positive mixed strategy. When a matrix game is stochastically extended with respect to at least one of the players; i.e., whenever one of the players' strategies are in fact mixtures of available pure strategies, then a mixed strategy for that player is said to be positive if it puts mass on all of the player's pure strategies. Thus, when the player, say the fighter, has a finite number of pure strategies, say $\sigma_1, \sigma_2, \cdots, \sigma_i, \cdots, \sigma_m$, as the fighter would in a finite version of the duel, then the fighter's strategy ξ is positive if ξ_i exceeds zero for all i, $i = 1, 2, \cdots, m$. Analogously, when the fighter has a continuum of pure strategies, as in the case of the duel studied in this thesis, then a mixed strategy for the fighter, as given by a cumulative distribution function $\sigma(p)$ on [c, R], is said to be positive if no non-degenerate sub-interval of [c, R] is assigned zero probability under σ.

When a game is stochastically extended with respect to one of the players, say with respect to the fighter, by the introduction of mixed strategies as above, and the fighter possesses a positive saddle point coordinate strategy, say σ_O^+, then any saddle point coordinate strategy τ_O attains the saddle point payoff v in the presence of all (pure or mixed) strategies for the fighter if, with r indexing the fighter's pure strategies, $M(r,\tau_O)$ is continuous in r. This is shown by noting first that, if $g(r)$ is non-positive and continuous,

$$\left[\int g(r)\,d\sigma_O^+(r) = 0\right] \text{ implies } [g(r) = 0]. \tag{3.1}$$

Then $[M(\sigma_O^+,\tau_O) - v = 0]$ implies $[\int M(r,\tau_O)\,d\sigma_O^+(r) - \int v\,d\sigma_O^+(r) = 0]$ implies $[\int [M(r,\tau_O) - v]\,d\sigma_O^+(r) = 0]$, which implies that $M(r,\tau_O) - v = 0$ by setting $g(r) = M(r,\tau_O) - v$ in Equation 3.1.

At any rate, then, if the fighter does possess a positive saddle point coordinate strategy, then, given the required continuity, it must be that any τ_O will satisfy

$$M(r,\tau_O) = v,\ \forall r, \tag{3.2}$$

which should allow us to compute a candidate τ_O. This feature motivates us to start looking for a saddle point (σ_O,τ_O) by searching for a τ_O satisfying Equation 3.2. If that search is successful, σ_O is then hunted down by looking for a σ such that τ_O minimizes M in the presence of σ; i.e., a σ such that

$$M(\sigma,\tau_O) \le M(\sigma,\tau),\ \forall \tau. \tag{3.3}$$

If the search for σ satisfying Equation 3.3 also is successful, and leads, say, to $\sigma = \sigma_O$, then (σ_O,τ_O) is established as a saddle point, since we already have by Equation 3.2 that

$$M(\sigma_O,\tau_O) = M(\sigma,\tau_O),\ \forall \sigma.$$

In the situation at hand (restricting r to the interval $[c, r_O]$, with the hope that things will fall into place by themselves on $[r_O, R]$), relation Equation 3.2 gives

$$F(r)e^{-\int_r^{r_O} p(s)\tau_O(s)\,ds} = v$$

$$\ln F(r) - \int_r^{r_O} p(s)\tau_O(s)\,ds = \ln v$$

$$\frac{F'(r)}{F(r)} + p(r)\tau_O(r) = 0$$

$$\tau_O(r) = \frac{-F'(r)}{F(r)p(r)} \equiv \tau^O(r).$$

We now define a candidate τ_O by extending τ_O to [c, R] by postulating that $\tau_O(r) = 0$, $r_O \le r \le R$, and apply Lagrangian saddle point methods to finding a σ_O such that

$$M(\sigma_O, \tau_O) \le M(\sigma_O, \tau).$$

Optimal Fighter Strategy by Lagrangian Saddle Point

In this section we will discuss how the fighter's optimal strategy ($\sigma^O(r)$) is derived using Lagrangian saddle point methods.

Our task is to find a σ such that

$$M(\sigma, \tau) - M(\sigma, \tau_O) \ge 0.$$

In other words, among the problems $P\sigma$

$$\begin{aligned} &Min && M(\sigma, \tau) \\ &\tau \text{ s.t.} && \int_c^R \tau(x)\,dx \le A \\ & && \tau \ge 0 \qquad \text{on } [\, c, R \,] \end{aligned}$$

parametrized by σ, find one, say P_{σ^O}, for which τ^O is the minimizing τ:

$$Min \quad M(\sigma^O, \tau) \quad = M(\sigma^O, \tau^O) \tag{3.4}$$

$$\tau \text{ s.t. } \quad \int_c^R \tau(x)\,dx \leq A$$
$$\tau \geq 0 \qquad \text{on } [c, R]$$

But the Lagrangian saddle point theory alerts us to the fact that it is sufficient for Equation 3.4 that τ^o participate in a saddle point of the Lagrangian $L(\lambda, \tau)$ for the minimization problem P_{σ^o} ; i.e., that there be a $\lambda_o \geq 0$ such that (λ_o, τ^o) is a saddle point of

$$L(\lambda, \tau) = M(\sigma^o, \tau) + \lambda\left[\int_c^R \tau(x)\,dx - A\right],$$

for all $\lambda \geq 0$ and $\tau \geq 0$ on $[c, R]$.

Thus σ^o will satisfy Equation 3.4 if there is a $\lambda_o \geq 0$ such that

$$\begin{aligned}
& M(\sigma^o, \tau) + \lambda_o\left[\int_c^R \tau(x)\,dx - A\right] \\
&\equiv \int_c^R \left[F(r)e^{-\int_r^R p(s)\tau(s)\,ds}\right] d\sigma^o(r) + \lambda_o\left[\int_c^R \tau(x)\,dx - A\right] \\
&\geq M(\sigma^o, \tau^o) + \lambda_o\left[\int_c^R \tau^o(x)\,dx - A\right] \qquad (3.5) \\
&\equiv \int_c^R F(r)e^{-\int_r^R p(s)\tau^o(s)\,ds}\, d\sigma^o(r) + \lambda_o\left[\int_c^R \tau^o(x)\,dx - A\right] \\
&\geq \int_c^R F(r)e^{-\int_r^R p(s)\tau^o(s)\,ds}\, d\sigma^o(r) + \lambda\left[\int_c^R \tau^o(x)\,dx - A\right],
\end{aligned}$$

for all $\lambda \geq 0$ and $\tau \geq 0$ on $[c, R]$. But, since

$$\int_c^R \tau^o(x)\,dx = A,$$

the second inequality is automatically satisfied with equality.

We can change the order of integration when Stieltjes integration is involved. As expected from the usual calculus, for non-negative A(.) and B(.) we have

$$\int_{r=c}^R B(r) \int_{x=r}^R A(x)\,dx d\sigma(r) = \int_{x=c}^R A(x) \int_{r=c}^x B(r)\,d\sigma(r) dx.$$

$$\epsilon^{x} \geq 1 + x.$$

$$\epsilon^{-z-z_O} - 1 \geq -(z - z_O).$$

$$\epsilon^{-z} - \epsilon^{-z_O} \geq -(z - z_O)\epsilon^{-z_O}.$$

$$\epsilon^{-\int_r^R a(s)\,ds} - \epsilon^{-\int_r^R a_O(s)\,ds} \geq -[\epsilon^{-\int_r^R a_O(s)\,ds}][\int_r^R (a(x) - a_O(x))\,dx].$$

$$\int_c^R F(r)\epsilon^{-\int_r^R a(s)\,ds}\,d\sigma(r) - \int_c^R F(r)\epsilon^{-\int_r^R a_O(s)\,ds}\,d\sigma(r) \tag{3.6}$$

$$\geq -\int_c^R [F(r)\epsilon^{-\int_r^R a_O(s)\,ds}][\int_r^R (a(x) - a_O(x))\,dx]\,d\sigma(r)$$

$$= -\int_c^R (a(x) - a_O(x))[\int_c^x F(r)\epsilon^{-\int_r^R a_O(s)\,ds}\,d\sigma(r)]\,dx.$$

where $B(r) = F(r)\epsilon^{-\int_r^R a_O(s)\,ds}$ and $A(x) = (a(x) - a_O(x))$.

Equation 3.6 yields the following equations when $\sigma(r) = \sigma^O(r)$, and $a_O(x) = p(x)\tau^O(x)$, $a(x) = p(x)\tau(x)$, with $\tau(x) \geq 0$ an arbitrary firing schedule for the bomber over [c,R], with $\int_c^R \tau(x)\,dx \leq A$.

$$\int_c^R [F(r)\epsilon^{-\int_r^R p(s)\tau(s)\,ds}]\,d\sigma^O(r) + \lambda_O[\int_c^R \tau(x)\,dx - A]$$

$$-\int_c^R F(r)\epsilon^{-\int_r^R p(s)\tau^O(s)\,ds}\,d\sigma^O(r) + \lambda_O[\int_c^R \tau^O(x)\,dx - A]$$

$$\geq \int_c^R [\tau(x) - \tau^O(x)][-p(x)\int_c^x F(r)\epsilon^{-\int_r^R p(s)\tau^O(s)\,ds}\,d\sigma^O(r)]\,dx$$

$$+\lambda_O[\int_c^R \tau(x) - \tau^O(x)]\,dx$$

$$= \int_c^R [\tau(x) - \tau^O(x)][-p(x)\int_c^x F(r)\epsilon^{-\int_r^R p(s)\tau^O(s)\,ds}\,d\sigma^O(r)$$

$$+\lambda_O]\,dx, \tag{3.7}$$

where, for later reference, we define

$$C'(x) = -p(x)\int_c^x F(r)e^{-\int_r^R p(s)\tau^o(s)\,ds}\,d\sigma^o(r) + \lambda_o.$$

Now we will examine the right hand side of Equation 3.7. The inequality of Equation 3.5 will be ensured by any σ^o and λ_o reducing C(x) to a function equal to zero on $[c, r_o]$ and greater than zero on $[r_o, R]$, say,

$$\sigma^o(x) \equiv \begin{cases} \dfrac{p(r_o)}{p(x)} & \text{on } [c, r_o] \\ 1 & \text{on } [r_o, R]. \end{cases}$$

and

$$\lambda_o = \lambda^o \equiv F(r_o)p(r_o) \geq 0,$$

since, with $\lambda_o = \lambda^o$, and repeating in part computations already done above, C'(x) becomes, for $x \leq r_o$,

$$\begin{aligned} C'(x) &= -p(x)\int_c^x F(r)e^{-\int_r^{r_o} p(s)\tau^o(s)\,ds}\,d\sigma^o(r) - \lambda^o \\ &= -p(x)\int_c^x F(r)\left[\frac{F(r_o)}{F(r)}\right]d\sigma^o(r) + \lambda^o \\ &= -p(x)F(r_o)\sigma^o(x) + F(r_o)p(r_o) \\ &= -p(x)F(r_o)\left[\frac{p(r_o)}{p(x)}\right] + F(r_o)p(r_o) \\ &= -F(r_o)p(r_o) + F(r_o)p(r_o) = 0, \end{aligned}$$

and, for $x \geq r_o$,

$$\begin{aligned} C'(x) &= -p(x)\int_c^{r_o} F(r)e^{-\int_r^R p(s)\tau^o(s)\,ds}\,d\sigma^o(r) \\ &\quad -p(x)\int_{r_o}^x F(r)e^{-\int_r^R p(s)\tau^o(s)\,ds}\,d\sigma^o(r) + \lambda^o \\ &= -p(x)\int_c^R F(r)e^{-\int_r^{r_o} p(s)\tau^o(s)\,ds}\,d\sigma^o(r) + \lambda^o \end{aligned}$$

$$= \quad -p(x)F(r_O)\sigma^O(r_O) + \lambda^O$$

$$= \quad -F(r_O)p(r_O) + F(r_O)p(r_O) \geq 0.$$

Since $C(x)$ is greater than or equal to zero, the right-hand side of Equation 3.7 is greater than or equal to zero. Therefore, the fighter's optimal strategy σ^O is found such that

$$M(\sigma^O, \tau) - M(\sigma^O, \tau^O) \geq 0. \tag{3.8}$$

Verification of Candidate Bomber Strategy

It remains to verify that the extended bomber strategy τ^O does in fact satisfy.

$$M(\sigma^O, \tau^O) \geq M(\sigma, \tau^O).$$

To this end note that

$$\begin{aligned}
M(\sigma^O, \tau^O) &= \int_C^R F(r) e^{-\int_r^R \tau^O(s)p(s)\,ds}\, d\sigma^O(r) \\
&= \int_C^{r_O} F(r) e^{-\int_r^{r_O} \tau^O(s)p(s)\,ds}\, d\sigma^O(r) \\
&= \int_C^{r_O} F(r) e^{\int_r^{r_O} \frac{F'(s)}{F(s)}\,ds}\, d\sigma^O(r) \\
&= \int_C^{r_O} F(r)\left[\frac{F(r_O)}{F(r)}\right] d\sigma^O(r) \\
&= F(r_O)\int_C^{r_O} d\sigma^O(r) \\
&= F(r_O),
\end{aligned}$$

where $\int_C^{r_O} d\sigma^O(r) = 1$.

$$M(\sigma, \tau^O) \quad = \quad \int_C^R F(r) e^{-\int_r^R \tau^O(s)p(s)\,ds}\, d\sigma(r)$$

$$
\begin{aligned}
&= \int_c^{r_o} F(r)e^{-\int_r^R \tau^o(s)p(s)\,ds}\,d\sigma(r) \\
&\quad + \int_{r_o}^{R} F(r)e^{-\int_r^R \tau^o(s)p(s)\,ds}\,d\sigma(r) \\
&= \int_c^{r_o} F(r)\left[\frac{F(r_o)}{F(r)}\right]d\sigma(r) + \int_{r_o}^{R} F(r)\,d\sigma(r) \\
&\leq F(r_o)\int_c^{r_o} d\sigma(r) + F(r_o)\int_{r_o}^{R} d\sigma(r) \\
&= F(r_o)\sigma(R) \leq F(r_o) = M(\sigma^o, \tau^o),
\end{aligned}
$$

where the last equality follows from $\sigma^o(r_o) = 1$.

Therefore, $M(\sigma^o, \tau^o) \geq M(\sigma, \tau^o)$, which, together with relation of Equation 3.8, verifies that σ^o and τ^o are the saddle point coordinate strategies for the fighter and bomber respectively.

Early Duel Start, with Altered Restriction on the Bomber

Consider any s with $c \leq s < r_o$. Suppose that the set of allowed strategies $\tau(r)$ for the bomber satisfy only the condition,

$$
\int_s^R \tau(r)\,dr \leq \int_s^R \tau^o(r)\,dr;
$$

i.e., that the bomber spend no more than the optimal amount in some early stage of the duel. Then a saddle point of the duel is given by the pair (σ_s^o, τ^o), where

$$
\sigma_s^o(r) := \begin{cases} 0 & \text{for } c \leq r < s, \\ \dfrac{p(r_o)}{p(r)} & \text{for } s \leq r < r_o, \\ 1 & \text{for } r \geq r_o. \end{cases}
$$

and

$$\tau^o(r) : \begin{cases} \dfrac{-F'(r)}{F(r)p(r)} & \text{for } c \le r < r_o, \\ 0 & \text{for } r \ge r_o. \end{cases}$$

To verify that $M(\sigma, \tau^o) \le M(\sigma_S^o, \tau^o)$, we can proceed as in the case of the classical fighter-bomber duel.

As to verifying that $M(\sigma_S^o, \tau) \ge M(\sigma_S^o, \tau^o)$, we proceed as follows:

From Equation 3.6,

$$\begin{aligned} &M(\sigma_S^o, \tau) - M(\sigma_S^o, \tau^o) \\ &= -\int_c^R (a(x) - a_o(x))[\int_c^x F(r)e^{-\int_r^R a_o(s)\,ds}\, d\sigma_S^o(r)]\, dx. \end{aligned} \tag{3.9}$$

where $a_o(x) = p(x)\tau^o(x)$ and $a(x) = p(x)\tau(x)$, with $\tau(x)$ an arbitrary firing schedule for the bomber over $[c, R]$.

To begin with, for $x \ge r_o$,

$$\begin{aligned} &\int_c^x [F(r)e^{-\int_r^R p(s)\tau^o(s)\,ds}]\, d\sigma_S^o(r) \\ &= \int_c^{r_o} F(r)e^{-\int_r^R p(s)\tau^o(s)\,ds}\, d\sigma_S^o(r) \\ &= \int_c^{r_o} F(r)e^{-\int_r^{r_o} p(s)\tau^o(s)\,ds}\, d\sigma_S^o(r) \\ &= \int_c^{r_o} F(r)e^{\int_r^{r_o} \frac{F'(s)}{F(s)}\,ds}\, d\sigma_S^o(r) \\ &= \int_c^{r_o} F(r)[\frac{F(r_o)}{F(r)}]\, d\sigma_S^o(r) \\ &= F(r_o)\int_c^{r_o} d\sigma_S^o(r) = F(r_o). \end{aligned}$$

And also, for $c \le x \le r_o$,

$$\int_c^x [F(r)e^{-\int_r^R p(s)\tau^o(s)\,ds}]\, d\sigma_S^o(r)$$

$$= \int_c^s 0\, d\sigma_s^o(r) + \int_s^x F(r) e^{-\int_r^{r_o} p(s)\tau^o(s)\, ds}\, d\sigma_s^o(r)$$

$$= \int_s^x F(r) e^{\int_r^{r_o} \frac{F'(s)}{F(s)}\, ds}\, d\sigma_s^o(r)$$

$$= \int_s^x F(r) [\frac{F(r_o)}{F(r)}]\, d\sigma_s^o(r)$$

$$= F(r_o) \int_s^x d\sigma_s^o(r) = F(r_o)\sigma_s^o(x).$$

And so,

$$\int_c^R [a(x) - a_o(x)][\int_c^x F(r) e^{-\int_r^R a_o(s)\, ds}\, d\sigma_s^o(r)]\, dx$$

$$\begin{aligned}
= &\int_c^{r_o} [a(x) - a_o(x)][\int_c^x F(r) e^{-\int_r^R a_o(s)\, ds}\, d\sigma_s^o(r)]\, dx \\
&+ \int_{r_o}^R [a(x) - a_o(x)][\int_c^x F(r) e^{-\int_r^R a_o(s)\, ds}\, d\sigma_s^o(r)]\, dx \\
= &\int_c^{r_o} [a(x) - a_o(x)][F(r_o)\sigma_s^o(x)]\, dx \\
&+ \int_{r_o}^R [a(x) - a_o(x)][F(r_o)]\, dx \qquad (3.10) \\
= &\int_c^s 0\, dx + \int_s^{r_o} [a(x) - a_o(x)][F(r_o)\sigma_s^o(x)]\, dx \\
&+ \int_{r_o}^R [a(x) - a_o(x)][F(r_o)]\, dx \\
\leq &\int_s^{r_o} [a(x) - a_o(x)][F(r_o)\frac{p(r_o)}{p(x)}]\, dx \\
&+ \int_{r_o}^R [a(x) - a_o(x)][F(r_o)\frac{p(r_o)}{p(x)}]\, dx
\end{aligned}$$

where this last inequality is due to the fact that $[a(x) - a_o(x)]$ is non-negative on $[r_o, R]$,

$$= \int_s^R [\tau(x) - \tau^o(x)][F(r_o)p(r_o)]\, dx$$

$$= [F(r_o)p(r_o)][\int_s^R \tau(x)\,dx - \int_s^R \tau^o(x)\,dx] \leq 0.$$

Hence, the right hand side of Equation 3.9 is greater than or equal to zero.

That in fact shows that

$$M(\sigma_S^o, \tau) - M(\sigma_S^o, \tau^o) \geq 0$$

Therefore, σ_S^o and τ^o are the saddle point coordinate strategies for the fighter and bomber respectively for our modified model.

Late Duel Start, with Initial Burst Possibility for the Bomber

As in earlier sections, let A and r_o be defined by

$$\int_c^{r_o} \frac{-F'(r)}{F(r)p(r)}\,dr = A.$$

We now suppose that the duel begins at a range $R_1 < r_o$. We also suppose that the bomber is capable of one ammunition burst at the range R_1, and that the fighter needs to survive any such initial burst at the range R_1 to execute any planned missile release at the range R_1.

Define

$$a_o = \int_{R_1}^{r_o} \frac{-F'(r)}{F(r)p(r)}\,dr.$$

The objective function for this model is as follows:

$$M(\sigma, \tau) = \int_c^{R_1} F(r)e^{-\int_r^{R_1} \tau(s)p(s)\,ds - p(R_1)a}\,d\sigma(r),$$

where $a = \int_{R_1}^{r_o} \tau(r)\,dr$.

Then we show below that the saddle point coordinate strategies are the natural ones that so-to-speak make up for lost time at the range R_1:

$$\sigma^o(r) : \begin{cases} \frac{p(r_o)}{p(r)} & \text{for } c \le r < R_1, \\ 1 & \text{for } r = R_1, \end{cases}$$

and

$$\tau^o(r) : \begin{cases} \frac{-F'(r)}{F(r)\bar{p}(r)} & \text{for } c \le r < R_1, \\ a_o & \text{for } r = R_1. \end{cases}$$

As to verifying that $M(\sigma^o, \tau) \ge M(\sigma^o, \tau^o)$, we proceed as follows:

From Equation 3.6,

$$\begin{aligned} & M(\sigma^o, \tau) - M(\sigma^o, \tau^o) \\ = & \int_c^{R_1} F(r) e^{-\int_r^{R_1} \tau(s)p(s)\,ds - p(R_1)a}\, d\sigma^o(r) \\ & - \int_c^{R_1} F(r) e^{-\int_r^{R_1} \tau^o(s)p(s)\,ds - p(R_1)a_o}\, d\sigma^o(r) \\ \ge & -\int_c^{R_1} [F(r) e^{-\int_r^{R_1} \tau^o(s)p(s)\,ds - p(R_1)a_o}] \\ & [\int_r^{R_1} (\tau(x)p(x) - \tau^o(x)p(x))\,dx + (a - a_o)p(R_1)]\, d\sigma^o(r) \\ = & -\alpha \int_c^{R_1} [\tau(x)p(x) - \tau^o(x)p(x)][\int_c^x F(r) e^{-\int_r^{R_1} \tau^o(s)p(s)\,ds}\, d\sigma^o(r)]\, dx \\ & -\alpha(a - a_o)p(R_1)[\int_c^{R_1} F(r) e^{-\int_r^{R_1} \tau^o(s)p(s)\,ds}\, d\sigma^o(r)] \end{aligned} \tag{3.11}$$

where $\alpha = e^{-p(R_1)a_o}$.

For $c \le x \le R_1$,

$$\begin{aligned} & \int_c^x F(r) e^{-\int_r^{R_1} \tau^o(s)p(s)\,ds}\, d\sigma^o(r) \\ = & \int_c^x [F(r) e^{-\int_r^{R_1} \tau^o(s)p(s)\,ds}]\, d\sigma^o(r) \end{aligned}$$

$$= \int_c^x F(r) e^{\int_r^{R_1} \frac{F'(s)}{F(s)}\, ds}\, d\sigma^o(r)$$

$$= \int_c^x F(r) [\frac{F(R_1)}{F(r)}]\, d\sigma^o(r)$$

$$= F(R_1) \int_c^x d\sigma^o(r)$$

$$= F(R_1) \sigma^o(x).$$

Also,

$$\int_c^{R_1} F(r) e^{-\int_r^{R_1} \tau^o(s) p(s)\, ds}\, d\sigma^o(r)$$

$$= \int_c^{R_1} F(r) e^{-\int_r^{R_1} \tau^o(s) p(s)\, ds}\, d\sigma^o(r)$$

$$= \int_c^{R_1} F(r) e^{\int_r^{R_1} \frac{F'(s)}{F(s)}\, ds}\, d\sigma^o(r)$$

$$= \int_c^{R_1} F(r) [\frac{F(R_1)}{F(r)}]\, d\sigma^o(r)$$

$$= F(R_1) \int_c^{R_1} d\sigma^o(r)$$

$$= F(R_1).$$

Hence the right hand side of Equation 3.11 equals,

$$-\alpha \int_c^{R_1} [\tau(x)p(x) - \tau^o(x)p(x)][F(R_1)\sigma^o(x)]\, dx - [\alpha(a - a_o)p(R1)F(R_1)]$$

$$= -\alpha\{ \int_c^{R_1} [\tau(x)p(x) - \tau^o(x)p(x)][F(R_1)\frac{p(R_1)}{p(x)}]\, dx + [(a - a_o)p(R_1)F(R_1)]\}$$

$$= \alpha F(R_1)p(R_1)[\int_c^{R_1} \tau^o(x)\, dx - \int_c^{R_1} \tau(x)\, dx + (a_o - a)]$$

$$= \alpha F(R_1)p(R_1)[(A - a_o) - (A' - a) + (a_o - a)]$$

$$= \alpha F(R_1)p(R_1)[A - A'] \geq 0$$

where A is the total amount of ammunition used under $\tau^o(r)$ and A' is the total amount of ammunition used under the arbitrary firing schedule $\tau(r)$ for the bomber.

That in fact shows that

$$M(\sigma^o, \tau) - M(\sigma^o, \tau^o) \geq 0.$$

It remains to show that

$$M(\sigma^o, \tau^o) \geq M(\sigma, \tau^o).$$

To this end note that

$$\begin{aligned}
M(\sigma^o, \tau^o) &= \int_c^{R_1} F(r) e^{-\int_r^{R_1} \tau^o(s) p(s)\, ds} e^{-p(R_1) a_o}\, d\sigma^o(r) \\
&= \alpha \int_c^{R_1} F(r) e^{\int_r^{R_1} \frac{F'(s)}{F(s)}\, ds}\, d\sigma^o(r) \\
&= \alpha \int_c^{R_1} F(r) e^{[\ln F(s)]_r^{R_1}}\, d\sigma^o(r) \\
&= \alpha \int_c^{R_1} F(r) [\frac{F(R_1)}{F(r)}]\, d\sigma^o(r) \\
&= \alpha F(R_1) \int_c^{R_1} d\sigma^o(r) \\
&= \alpha F(R_1),
\end{aligned}$$

where $\int_c^{R_1} d\sigma^o(r) = 1$, and

$$\begin{aligned}
M(\sigma, \tau^o) &= \int_c^{R_1} F(r) e^{-\int_r^{R_1} \tau^o(s) p(s)\, ds} e^{-p(R_1) a_o}\, d\sigma(r) \\
&= \alpha \int_c^{R_1} F(r) e^{\int_r^{R_1} \frac{F'(s)}{F(s)}\, ds}\, d\sigma(r) \\
&= \alpha \int_c^{R_1} F(r) [\frac{F(R_1)}{F(r)}]\, d\sigma(r)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha F(R_1) \int_c^{R_1} d\sigma(r) \\
&= \alpha F(R_1)\sigma(R_1).
\end{aligned}
$$

Then,

$$
\begin{aligned}
M(\sigma^o, \tau^o) - M(\sigma, \tau^o) &= \alpha F(R_1) - \alpha F(R_1)\sigma(R_1) \\
&= \alpha F(R_1)(1 - \sigma(R_1)) \geq 0,
\end{aligned}
$$

since $\sigma(R_1) \leq 1$.

Hence, $M(\sigma^o, \tau^o) \geq M(\sigma, \tau^o)$.

Therefore, σ^o and τ^o are the saddle point coordinate strategies for the fighter and bomber respectively when the duel starts at a range R_1 less than the range r_o.

CHAPTER 4. ANALYSIS OF THE KARLIN MODEL

Introductory Remarks

This chapter examines the optimal strategies suggested by Karlin [23] for the fighter and bomber, under Karlin's chief distinguishing assumption that the bomber's firing rate is bounded. Also, we will use the notation for the Weiss–Gillman model in this chapter. We assume with Karlin that $F(r)$ and $p(r)$ decrease with range, and that $\frac{-F'(r)}{F(r)p(r)}$ is strictly increasing on range r and the minimum closing range c is equal to zero. Finally, we work with the assumption of the fourth section of Chapter 3, under which,

$$\int_s^R \tau(r)\,dr \leq \int_s^R \tau^O(r)\,dr.$$

Early Duel Start, with Altered Restriction on the Bomber

Consider any s with $c \leq s < r_O$. Suppose that the set of allowed strategies $\tau(r)$ for the bomber satisfy the above condition.

Let us define the following quantity and functions:

$$w(r) \qquad = \min\{1, \frac{-F'(r)}{F(r)p(r)}\}.$$

$$d \qquad = \min\{x : \frac{-F'(x)}{F(x)p(x)} \geq 1\}.$$

$$\int_0^1 w(r)\,dr \quad = \delta_o = \delta.$$

When $\delta_o = \delta$, $c < d$,

a saddle point coordinate strategy for the fighter is

$$\sigma_s^o(r) : \begin{cases} 0 & \text{for } 0 \le r < s, \\ \frac{p(r_o)}{p(r)} & \text{for } s \le r < r_o, \\ 1 & \text{for } r \ge r_o; \end{cases}$$

a saddle point coordinate strategy for the bomber is

$$\tau^o(r) : \begin{cases} w(r) & \text{for } 0 \le r < r_o, \\ 0 & \text{for } r \ge r_o. \end{cases}$$

When $\delta_o = \delta$, $0 < r_o \le d$, we can prove this in the same way as for the Weiss-Gillman model.

Now suppose that $0 \le d \le r_o$. We verify that $M(\sigma_s^o, \tau) \ge M(\sigma_s^o, \tau^o)$, using Equation 3.6, as follows:

$$\begin{aligned} & M(\sigma_s^o, \tau) - M(\sigma_s^o, \tau^o) \\ &= -\int_0^R (a(x) - a_o(x))\left[\int_0^x F(r) e^{-\int_r^R a_o(s)\,ds}\, d\sigma_s^o(r)\right] dx. \end{aligned} \tag{4.1}$$

where $a_o(x) = p(x)\tau^o(x)$ and $a(x) = p(x)\tau(x)$, with τ an arbitrary firing schedule for the bomber over $[c, R]$.

We will show that $M(\sigma_s^o, \tau) \ge M(\sigma_s^o, \tau^o)$, by showing that Equation 4.1 is greater than or equal to zero.

To begin with, for $x \ge r_o$,

$$\int_0^x \left[F(r) e^{-\int_r^R p(s)\tau^o(s)\,ds}\right] d\sigma_s^o(r)$$

$$\int_0^{r_o} [F(r)e^{-\int_r^{r_o} p(s)\tau^o(s)\,ds}]\, d\sigma_s^o(r)$$
$$= \int_0^{r_o} F(r)e^{[-\int_r^d p(s)\tau^o(s)\,ds - \int_d^{r_o} p(s)\,ds]}\, d\sigma_s^o(r)$$
$$\leq \int_0^s 0\, d\sigma_s^o(r) + \int_s^{r_o} F(r)e^{-\int_r^{r_o} p(s)\tau^o(s)\,ds}\, d\sigma_s^o(r)$$
$$= \int_s^{r_o} F(r)e^{\int_r^{r_o} \frac{F'(s)}{F(s)}\,ds}\, d\sigma_s^o(r)$$
$$= \int_s^{r_o} F(r)[\frac{F(r_o)}{F(r)}]\, d\sigma_s^o(r)$$
$$= F(r_o)\int_s^{r_o} d\sigma_s^o(r) = F(r_o),$$

since $\int_s^{r_o} d\sigma_s^o(r) = 1$. Note that $0 \leq \tau(r) \leq 1$ by assumption of the Karlin model. And for $0 \leq x \leq r_o$,

$$\int_0^{x} [F(r)e^{-\int_r^{r_o} p(s)\tau^o(s)\,ds}]\, d\sigma_s^o(r)$$
$$= \int_0^{x} F(r)e^{[-\int_r^d p(s)\tau^o(s)\,ds - \int_d^{r_o} p(s)\,ds]}\, d\sigma_s^o(r)$$
$$\leq \int_0^s 0\, d\sigma_s^o(r) + \int_s^{x} F(r)e^{-\int_r^{r_o} p(s)\tau^o(s)\,ds}\, d\sigma_s^o(r)$$
$$= \int_s^{x} F(r)e^{\int_r^{r_o} \frac{F'(s)}{F(s)}\,ds}\, d\sigma_s^o(r)$$
$$= \int_s^{x} F(r)[\frac{F(r_o)}{F(r)}]\, d\sigma_s^o(r)$$
$$= F(r_o)\int_s^{x} d\sigma_s^o(r) = F(r_o)\sigma_s^o(x).$$

And so,

$$\int_0^R [a(x) - a_o(x)][\int_0^x F(r)e^{-\int_r^R a_o(s)\,ds}\, d\sigma_s^o(r)]\, dx$$
$$= \int_0^{r_o} [a(x) - a_o(x)][\int_0^x F(r)e^{-\int_r^R a_o(s)\,ds}\, d\sigma_s^o(r)]\, dx$$

$$
\begin{aligned}
&+\int_{r_o}^{R}[a(x)-a_o(x)][\int_0^x F(r)e^{-\int_r^R a_o(s)\,ds}\,d\sigma_s^o(r)]\,dx\\
\le\ & \int_0^{r_o}[a(x)-a_o(x)][F(r_o)\sigma_s^o(x)]\,dx+\int_{r_o}^{R}[a(x)-a_o(x)][F(r_o)]\,dx\\
=\ & \int_0^{s}0\,dx+\int_s^{r_o}[a(x)-a_o(x)][F(r_o)\sigma_s^o(x)]\,dx+\int_{r_o}^{R}[a(x)-a_o(x)][F(r_o)]\,dx\\
\le\ & \int_s^{r_o}[a(x)-a_o(x)][F(r_o)\frac{p(r_o)}{p(x)}]\,dx+\int_{r_o}^{R}[a(x)-a_o(x)][F(r_o)\frac{p(r_o)}{p(x)}]\,dx\\
=\ & \int_s^{r_o}[\tau(x)-\tau^o(x)][F(r_o)p(r_o)]\,dx+\int_{r_o}^{R}[\tau(x)-\tau^o(x)][F(r_o)p(r_o)]\,dx\\
=\ & [F(r_o)p(r_o)][\int_s^{R}\tau(x)\,dx-\int_s^{R}\tau^o(x)\,dx]\le 0.
\end{aligned}
$$

Hence, the quantity of Equation 4.1 is greater than or equal to zero. That in fact proves that

$$M(\sigma_s^o,\tau)-M(\sigma_s^o,\tau^o)\ge 0.$$

It remains to show that

$$M(\sigma_s^o,\tau^o)\ge M(\sigma,\tau^o).$$

To begin with,

$$
\begin{aligned}
M(\sigma_s^o,\tau^o) &= \int_0^{R}F(r)e^{-\int_r^{r_o}\tau^o(s)p(s)\,ds}\,d\sigma_s^o(r)\\
&= \int_0^{r_o}F(r)e^{[-\int_r^d \frac{-F'(s)}{F(s)p(s)}p(s)\,ds-\int_d^{r_o}p(s)\,ds]}\,d\sigma_s^o(r)\\
&= \int_0^{r_o}F(r)e^{[\int_r^d \frac{F'(s)}{F(s)}\,ds]}e^{[-\int_d^{r_o}p(s)\,ds]}\,d\sigma_s^o(r)\\
&= \int_0^{s}0\,d\sigma_s^o(r)+\int_s^{r_o}F(r)p_o e^{[\ln F(s)]_r^d}\,d\sigma_s^o(r)\\
&= \int_s^{r_o}F(r)p_o[\frac{F(d)}{F(r)}]\,d\sigma_s^o(r)
\end{aligned}
$$

$$\begin{aligned}
&= F(d)p_O \int_s^{r_O} d\sigma_S^O(r) \\
&= F(d)p_O.
\end{aligned}$$

where $p_O = e^{-\int_d^{r_O} p(s)\,ds}$, and $\int_s^{r_O} d\sigma_S^O(r) = 1$.

Then,

$$\begin{aligned}
M(\sigma, \tau^O) &= \int_0^R F(r) e^{-\int_r^R \tau^O(s)p(s)\,ds}\, d\sigma(r) \\
&= \int_0^R F(r) e^{-\int_r^{r_O} \tau^O(s)p(s)\,ds}\, d\sigma(r) \\
&= \int_0^R F(r) e^{\left[\int_r^d \frac{F'(s)}{F(s)}\,ds\right]} e^{\left[-\int_d^{r_O} p(s)\,ds\right]}\, d\sigma(r) \\
&= \int_0^R F(r) p_O e^{\left[\ln F(s)\right]_r^d}\, d\sigma(r) \\
&= \int_0^R F(r) \left[\frac{F(d)p_O}{F(r)}\right] d\sigma(r) \\
&= F(d)p_O \int_0^R d\sigma(r) \\
&= F(d)p_O\sigma(R) \leq F(d)p_O = M(\sigma_S^O, \tau^O).
\end{aligned}$$

Therefore, $M(\sigma_S^O, \tau^O) \geq M(\sigma, \tau^O)$, and σ_S^O and τ^O are the saddle point coordinate strategies for the fighter and bomber for the Karlin model.

CHAPTER 5. LATE DUEL START WITH SPECIAL STRUCTURE

Integer Relationship between $F(r)$ and $G(r)$

This section derives a saddle point coordinate strategy for a late–start duel with special structure, for which an initial bomber firing burst is not allowed.

Suppose that there is a non–increasing differentiable positive function $G(r) \leq 1$ such that the fighter lethality function $F(r)$ has form

$$F(r) = 1 - (1 - G(r))^m,$$

with m an integer, and such that the duel begins at range r_g, where r_g is defined by

$$\int_c^{r_g} \frac{-G'(r)}{G(r)p(r)}\, dr = A.$$

We first study the relation between r_g and r_o defined by

$$\int_c^{r_o} \frac{-F'(r)}{F(r)p(r)}\, dr = A.$$

Indeed, we show that $r_g \leq r_o$ (which relation explains why we see this section as treating "late start").

When $m = 2$, $F(r) = 1 - (1 - G(r))^2$ and $F'(r) = 2G'(r)(1 - G(r))$. Hence,

$$\frac{-F'(r)}{F(r)p(r)} - (\frac{-G'(r)}{G(r)p(r)})$$
$$= (\frac{1}{p(r)})[\frac{-F'(r)}{F(r)} + \frac{G'(r)}{G(r)}]$$

$$= (\frac{1}{p(r)})[\frac{-2G'(r)(1-G(r))}{G(r)(2-G(r))} + \frac{G'(r)}{G(r)}]$$
$$= (\frac{1}{p(r)G(r)(2-G(r))})[-2G'(r)(1-G(r)) + G'(r)(2-G(r))]$$
$$= \frac{G'(r)G(r)}{p(r)(1-(1-G(r))^2)} \le 0,$$

since $G'(r) \le 0$, $G(r) > 0$, and $1 - (1 - G(r)) > 0$.

Therefore, $\frac{-F'(r)}{F(r)p(r)} \le \frac{-G'(r)}{G(r)p(r)}$, so that $r_g \le r_o$ for m = 2.

For $m > 2$, we focus attention on the term,

$$H(r) = \frac{-F'(r)}{F(r)} - (\frac{-G'(r)}{G(r)}),$$

where $F(r) = 1 - (1 - G(r))^m$, $m = 3, 4, \cdots, k$. The following results are obtained:

$m = 3:$
$$H(r) = (\frac{G'(r)G(r)}{1-(1-G(r))^3})[(1-G(r))3 + G(r)]$$

$m = 4:$
$$H(r) = (\frac{G'(r)G(r)}{1-(1-G(r))^4})[(1-G(r))[3(2-G(r))] + G(r)]$$

$m = 5:$
$$H(r) = (\frac{G'(r)G(r)}{1-(1-G(r))^5})[(1-G(r))[4(1-G(r))^2$$
$$+3(2-G(r))] + G(r)]$$

$m = 6:$
$$H(r) = (\frac{G'(r)G(r)}{1-(1-G(r))^6})[(1-G(r))[5(1-G(r))^3 + 4(1-G(r))^2$$
$$+3(2-G(r))] + G(r)]$$

$m = 7:$

$$H(r) = \left(\frac{G'(r)G(r)}{1-(1-G(r))^7}\right)[(1-G(r))[6(1-G(r))^4 + 5(1-G(r))^3$$
$$+4(1-G(r))^2 + 3(2-G(r))] + G(r)]$$

$m = k:$

$$H(r) = \left(\frac{G'(r)G(r)}{1-(1-G(r))^k}\right)[(1-G(r))[(k-1)(1-G(r))^{k-3}$$
$$+(k-2)(1-G(r))^{k-4} + \cdots + 4(1-G(r))^2 + 3(2-G(r))] + G(r)].$$

Therefore, we can conclude that

$$H(r) = \frac{-F'(r)}{F(r)} - \left(\frac{-G'(r)}{G(r)}\right) \leq 0$$

for all cases, since $G'(r) \leq 0$, $0 < G(r) \leq 1$, and $1-(1-G(r))^k > 0$, and $(1-G(r))[(k-1)(1-G(r))^{k-3} + (k-2)(1-G(r))^{k-4} + \cdots + 4(1-G(r))^2 + 3(2-G(r))] + G(r) \geq 0$, for all k.

Therefore, $\frac{-F'(r)}{F(r)p(r)} \leq \frac{-G'(r)}{G(r)p(r)}$, and $r_g \leq r_o$ for all k.

We now show that a saddle point coordinate strategy for the fighter is to fire at range r_g, and a saddle point coordinate strategy for the bomber is to use firing intensity

$$\tau^o(r): \begin{cases} \frac{-G'(r)}{G(r)p(r)} & \text{for } c \leq r < r_g, \\ 0 & \text{for } r \geq r_g. \end{cases}$$

We verify that $K(r_g, \tau) = K(r_g, \tau^o)$ as follows:

$$K(r_g, \tau) - K(r_g, \tau^o)$$
$$= F(r_g)e^{-\int_{rg}^{rg} \tau(s)p(s)\,ds} - F(r_g)e^{-\int_{rg}^{rg} \tau^o(s)p(s)\,ds}$$

$$= F(r_g) - F(r_g) = 0.$$

It remains to show that

$$K(r_g, \tau^o) \geq K(r, \tau^o) \text{ for } c \leq r \leq r_g.$$

To this end note that

$$\begin{aligned}
&K(r_g, \tau^o) - K(r, \tau^o) \\
&= F(r_g) e^{-\int_{r_g}^{r_g} \tau^o(s) p(s)\, ds} - F(r) e^{-\int_r^{r_g} \tau^o(s) p(s)\, ds} \\
&= F(r_g) - F(r) e^{-\int_r^{r_g} \tau^o(s) p(s)\, ds} \\
&= F(r_g) - F(r) e^{\int_r^{r_g} \frac{G'(s)}{G(s)}\, ds} \\
&= F(r_g) - F(r) \frac{G(r_g)}{G(r)} \\
&= [1 - (1 - G(r_g))^m] - [1 - (1 - G(r))^m] \frac{G(r_g)}{G(r)} \\
&= \frac{1 - (1 - G(r_g))^m}{G(r_g)} - \frac{1 - (1 - G(r))^m}{G(r)} \\
&\equiv \frac{1 - y^m}{1 - y} - \frac{1 - x^m}{1 - x} \geq 0,
\end{aligned}$$

since $0 \leq x \leq y < 1$, and with z set equal either to x or to y,

$$\frac{1 - z^m}{1 - z} = 1 + z + z^2 + z^3 + \cdots + z^{m-1}.$$

Real Relationship between $F(r)$ and $G(r)$

When m is generalized to an arbitrary real number $a > 1$, the saddle point of the previous section still obtains, and the analysis only changes in the demonstration of the fact that

$$K(r_g, \tau^o) \geq K(r, \tau^o).$$

To this end note that

$$
\begin{aligned}
&K(r_g, \tau^o) - K(r, \tau^o) \\
&= F(r_g)e^{-\int_{r_g}^{r_g} \tau^o(s)p(s)\,ds} - F(r)e^{-\int_r^{r_g} \tau^o(s)p(s)\,ds} \\
&= F(r_g) - F(r)e^{-\int_r^{r_g} \tau^o(s)p(s)\,ds} \\
&= F(r_g) - F(r)e^{\int_r^{r_g} \frac{G'(s)}{G(s)}\,ds} \\
&= F(r_g) - F(r)\frac{G(r_g)}{G(r)} \\
&= [1-(1-G(r_g))^a] - [1-(1-G(r))^a]\frac{G(r_g)}{G(r)} \\
&= \frac{1-(1-G(r_g))^a}{G(r_g)} - \frac{1-(1-G(r))^a}{G(r)} \\
&\equiv \frac{1-y^a}{1-y} - \frac{1-x^a}{1-x} \geq 0
\end{aligned}
$$

where $0 \leq x \leq y < 1$, and $G(r_g) \leq G(r)$, for $r \leq r_g$.

We now verify that $\frac{d}{dx}(\frac{1-x^a}{1-x})$ is greater than zero for x in the interval $(0 \leq x < 1)$:

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{1-x^a}{1-x}\right) &= \frac{-ax^{a-1}}{(1-x)} + \frac{(1-x^a)}{(1-x)^2} \\
&= \frac{-ax^{a-1} + ax^a + 1 - x^a}{(1-x)^2} \\
&= \frac{1}{(1-x)^2}[1 + x^{a-1}((a-1)x - a)]
\end{aligned}
$$

When $x = 0$, and a is any real number larger than one,

$$\frac{d}{dx}\left(\frac{1-x^a}{1-x}\right) = 1 > 0.$$

For $0 \leq x < 1$, we want to show that

$$\frac{d}{dx}\left(\frac{1-x^a}{1-x}\right) > 0,$$

by proving that $1 - x^{a-1}(ax - x - a) > 0$. But

$$\begin{aligned} 1 - x^{a-1}(ax - x - a) &> 0 \Longrightarrow \\ x^{a-1}(ax - x - a) &> -1 \Longleftarrow \\ ax - x - a &> -\frac{1}{x^{a-1}} \Longrightarrow \\ z_1(x) \equiv (1-a)x + a &< z_2(x) \equiv \frac{1}{x^{a-1}}. \end{aligned} \tag{5.1}$$

The inequality of Equation 5.1 which can be shown in Figure 5.1, is satisfied for all $a > 1$. Note that $z_1(x = 1) = 1$ and $z_2(x = 1) = 1$, and the slope of $z_1(x)$ is $(1 - a)$ in the interval $0 < x < 1$, and that of $z_2(x)$ is $(1 - a)$ at $x = 1$, and the slope of $z_2(x)$ is $(\frac{1-a}{x^a})$ larger than that of $z_1(x)$ for $0 < x < 1$.

Therefore, $K(r_g, \tau^o) \geq K(r, \tau^o)$, and r_g and τ^o are the saddle point coordinate strategies for the fighter and bomber for a late duel start model with special structure.

Extended Relationship between $F(r)$ and $G(r)$

If there is a non-increasing differentiable positive function $G(r) \leq 1$ such that

$$\int_c^{r_g} \frac{-G'(r)}{G(r)p(r)}\, dr = A,$$

and

$$\frac{F(r)}{F(r_g)} \leq \frac{G(r)}{G(r_g)}$$

over $c \leq r \leq r_g$, then the saddle point coordinate strategies for the fighter and bomber are the same as those of the previous section. Once again, the analysis differs only in the manner in which

$$K(r_g, \tau^o) \geq K(r, \tau^o)$$

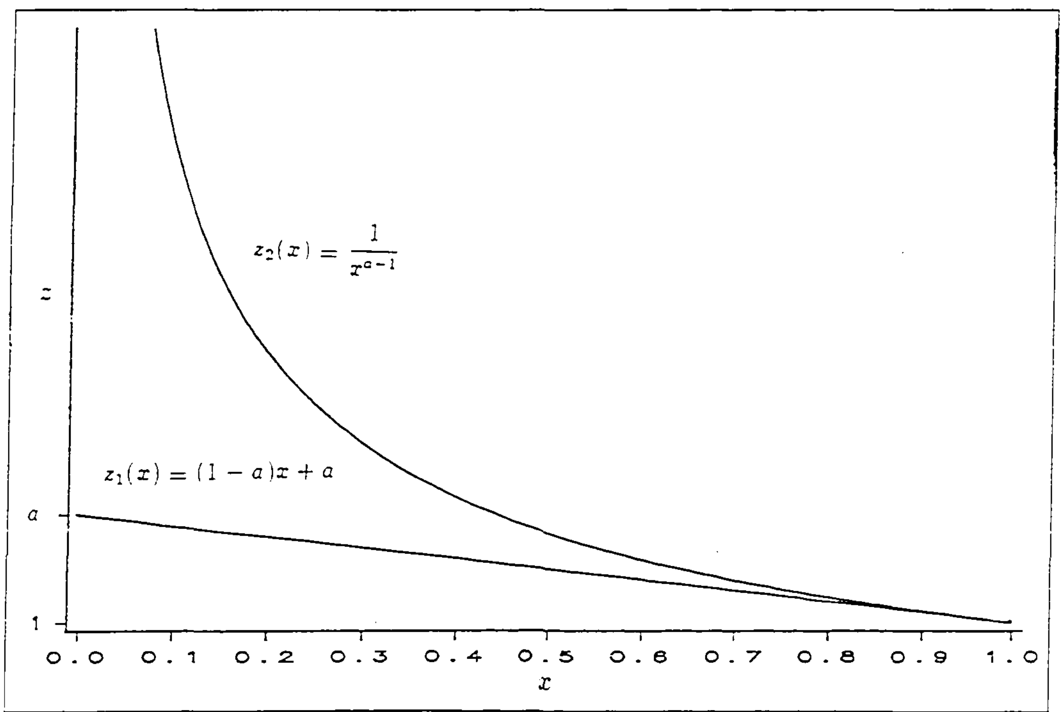

Figure 5.1: $z_1(x) < z_2(x)$

is to be verified.

To this end note that

$$
\begin{aligned}
&K(r_g, \tau^o) - K(r, \tau^o) \\
&= F(r_g)e^{-\int_{r_g}^{r_g} \tau^o(s)p(s)\,ds} - F(r)e^{-\int_r^{r_g} \tau^o(s)p(s)\,ds} \\
&= F(r_g) - F(r)e^{-\int_r^{r_g} \tau^o(s)p(s)\,ds} \\
&= F(r_g) - F(r)e^{\int_r^{r_g} \frac{G'(s)}{G(s)}\,ds} \\
&= F(r_g) - F(r)\frac{G(r_g)}{G(r)} \\
&= \frac{G(r)}{G(r_g)} - \frac{F(r)}{F(r_g)} \geq 0.
\end{aligned}
$$

since $\frac{F(r)}{F(r_g)} \leq \frac{G(r)}{G(r_g)}$ by assumption for $c \leq r \leq r_g$.

Therefore, $K(r_g, \tau^o) \geq K(r, \tau^o)$, and r_g and τ^o are the saddle point coordinate strategies for the fighter and bomber.

CHAPTER 6. FIGHTER WITH MULTIPLE MISSILES

Introductory Remarks

This chapter derives a saddle point coordinate strategy for the extension of the duel to the case where the fighter has multiple identical missiles. Let us suppose that a fighter has multiple missiles to attack a bomber instead of one missile in the classical fighter-bomber duel problem. Let us assume that the fighter can launch several missiles at the same time, or at the same ıange, or release one missile at range r_1 and then one missile at range r_2 and so on. Two extreme cases here would be that in which there is simultaneously launch; in other words, the case of one super–missile, and the case in which distinct ranges $r_1, r_2, r_3, \cdots r_n$ are involved. We assume that the duel starts at the range r_o determined by the single-missile lethality function $F(r)$, which then implies that the ranges r_1, $r_2, \cdots, r_n$ are less than or equal to the range r_o. The other conditions are the same as those of the classical fighter-bomber duel problem.

For this extension of the classical fighter-bomber duel, define the following variables.

I: A fighter with multiple missiles as Player one

II: A bomber as Player two, capable of continuous fire

r_i: Range for the fighter to launch the i-th missile.

$F(r_i)$: Lethality function for the fighter to fire at range r_i.

$\epsilon^{-\int_{r_1}^{r_o} \tau(s)p(s)\,ds}$: Probability that the fighter survives from range r_o to range r_1.

$\epsilon^{-\int_{r_2}^{r_1} \tau(s)p(s)\,ds}$: Probability that the fighter survives from range r_1 to range r_2.

$K((r_1, r_2, \cdots, r_n), \tau)$: Payoff in which the bomber uses firing intensity $\tau(r)$ and the fighter fires at ranges $r_1, r_2, \cdots, r_n$.

Fighter With 2 Missiles

Let us suppose that r_1, and r_2 are the ranges for the fighter to fire the first and second missiles respectively.

A saddle point coordinate strategy for the fighter is to fire the two missiles at range r_o.

A saddle point coordinate strategy for the bomber is

$$\tau^o(r) : \begin{cases} \dfrac{-F'(r)}{F(r)p(r)} & \text{for } c \le r < r_o, \\ 0 & \text{for } r \ge r_o. \end{cases}$$

To show this, note first that the objective function now is as follows:

$$K((r_1, r_2), \tau) = F(r_1)\epsilon^{-\int_{r_1}^{r_o} \tau(s)p(s)\,ds} + (1 - F(r_1))F(r_2)\epsilon^{-\int_{r_2}^{r_o} \tau(s)p(s)\,ds},$$

where we have assumed that the missile launch outcomes are statistically independent of each other. The total missile launch outcomes for the fighter which fires simultaneously two missiles at any range r is defined as follows:

$$T(r) = 1 - (1 - F(r))^2 = 2F(r) - (F(r))^2.$$

When both players adopt the saddle point coordinate strategies for each one, the probability that the bomber is killed is computed as

$$
\begin{aligned}
&K((r_o, r_o), \tau^o) \\
&= F(r_o)e^{-\int_{r_o}^{r_o} \tau^o(s)p(s)\,ds} + (1 - F(r_o))F(r_o)e^{-\int_{r_o}^{r_o} \tau^o(s)p(s)\,ds} \\
&= F(r_o) + (1 - F(r_o))F(r_o) \\
&= 2F(r_o) - (F(r_o))^2,
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
K((r_o, r_o), \tau^o) &= T(r_o)e^{-\int_{r_o}^{r_o} \tau^o(s)p(s)\,ds} \\
&= T(r_o) = 2F(r_o) - (F(r_o))^2.
\end{aligned}
$$

Now we prove that $K((r_o, r_o), \tau) = K((r_o, r_o), \tau^o)$:

$$
\begin{aligned}
&K((r_o, r_o), \tau) - K((r_o, r_o), \tau^o) \\
&= T(r_o)e^{-\int_{r_o}^{r_o} \tau(s)p(s)\,ds} - T(r_o)e^{-\int_{r_o}^{r_o} \tau^o(s)p(s)\,ds} \\
&= T(r_o) - T(r_o) = 0.
\end{aligned}
$$

Then we show that $K((r_o, r_o), \tau^o) \geq K((r_1, r_2), \tau^o)$:

$$
\begin{aligned}
&K((r_o, r_o), \tau^o) - K((r_1, r_2), \tau^o) \\
= \; & T(r_o) - [F(r_1)e^{-\int_{r_1}^{r_o} \tau^o(s)p(s)\,ds} \\
& +(1 - F(r_1))F(r_2)e^{-\int_{r_2}^{r_o} \tau^o(s)p(s)\,ds}] \\
= \; & T(r_o) - [F(r_1)e^{\int_{r_1}^{r_o} \frac{F'(s)}{F(s)}\,ds} \\
& +(1 - F(r_1))F(r_2)e^{\int_{r_2}^{r_o} \frac{F'(s)}{F(s)}\,ds}]
\end{aligned}
$$

$$
\begin{aligned}
&= T(r_o) - [F(r_1)\frac{F(r_o)}{F(r_1)} + (1 - F(r_1))F(r_2)\frac{F(r_o)}{F(r_2)}] \\
&= T(r_o) - F(r_o)(2 - F(r_1)) \\
&= F(r_o)(2 - F(r_o)) - F(r_o)(2 - F(r_1)) \geq 0,
\end{aligned}
$$

since $F(r_o) \leq F(r_1)$, and $(2 - F(r_o)) \geq (2 - F(r_1))$.

Therefore, $K((r_o, r_o), \tau^o)$ satisfies the required minimum and maximum conditions, and (r_o, r_o) and τ^o are the saddle point coordinate strategies for extension of the duel in which the fighter has two missiles.

Fighter With n Missiles

Suppose that a fighter has n number of missiles to attack a bomber, and the fighter can launch the missiles at the same time, at some range r, or can launch the missiles at n different ranges. Suppose once again that the duel begins at range r_o, determined by the lethality function $F(r)$ of a single missile.

The objective function is as follows:

$$
\begin{aligned}
& K((r_1, r_2, \cdots, r_n), \tau) \\
= \; & F(r_1)e^{-\int_{r_1}^{r_o} \tau(s)p(s)\,ds} + (1 - F(r_1))F(r_2)e^{-\int_{r_2}^{r_o} \tau(s)p(s)\,ds} \\
& + \cdots + (1 - F(r_1))(1 - F(r_2)) \cdots (1 - F(r_{n-2})) \\
& F(r_{n-1})e^{-\int_{r_{n-1}}^{r_o} \tau(s)p(s)\,ds} + (1 - F(r_1))(1 - F(r_2)) \cdots \\
& (1 - F(r_{n-1}))F(r_n)e^{-\int_{r_n}^{r_o} \tau(s)p(s)\,ds},
\end{aligned}
$$

where we have also assumed that the missile launch outcomes are statistically independent of each other for this case as in the previous section.

A saddle point coordinate strategy for the fighter is to fire the n missiles at range r_O.

A saddle point coordinate strategy for the bomber is

$$\tau^O(r): \begin{cases} \dfrac{-F'(r)}{F(r)p(r)} & \text{for } c \leq r < r_O, \\ 0 & \text{for } r \geq r_O. \end{cases}$$

When the fighter has two missiles (n=2), we proved in the previous section that (r_O, r_O) and τ^O are the saddle point coordinate strategies. Now we will show the solution specified in the section is true for $n \geq 2$ and look at n missile case. The total missile launch outcomes for the fighter which fires n number of identical missiles at the same range r is defined as

$$T(r) = 1 - (1 - F(r))^n.$$

When both players adopt the saddle point coordinate strategies for each one, the probability that the bomber is killed is computed as

$$\begin{aligned} K((r_O, r_O, \cdots, r_O), \tau^O) &= T(r_O)e^{-\int_{r_O}^{r_O} \tau^O(s)p(s)\,ds} \\ &= T(r_O) = 1 - (1 - F(r_O))^n. \end{aligned}$$

We show that $K((r_O, r_O, \cdots, r_O), \tau) = K((r_O, r_O, \ldots, r_O), \tau^O)$ when the fighter has n missiles.

$$\begin{aligned} &K((r_O, r_O, \cdots, r_O), \tau) - K((r_O, r_O, \cdots, r_O), \tau^O) \\ &= T(r_O)e^{-\int_{r_O}^{r_O} \tau(s)p(s)\,ds} - T(r_O)e^{-\int_{r_O}^{r_O} \tau^O(s)p(s)\,ds} \\ &= T(r_O) - T(r_O) = 0. \end{aligned}$$

Then we prove that $K((r_O, r_O, \cdots, r_O), \tau^O) \geq K((r_1, r_2, \cdots, r_n), \tau^O)$. Note that $F(r_O) \leq F(r_1) \leq F(r_2) \leq \cdots \leq F(r_n)$ because $r_O \geq r_1 \geq r_2 \geq \cdots \geq r_n \geq c$.

We start out looking at the case $n = 3$.

When the fighter has three missiles,

$$
\begin{aligned}
K((r_o, r_o, r_o), \tau^o) &= [1 - (1 - F(r_o))^3] e^{-\int_{r_o}^{r_o} \tau^o(s) p(s)\, ds} \\
&= 1 - (1 - F(r_o))^3 \\
&= 3F(r_o) - 3F(r_o)^2 + F(r_o)^3.
\end{aligned}
$$

And also,

$$
\begin{aligned}
K((r_1, r_2, r_3), \tau^o) &= F(r_1) e^{-\int_{r_1}^{r_o} \tau^o(s) p(s)\, ds} \\
&\quad + (1 - F(r_1)) F(r_2) e^{-\int_{r_2}^{r_o} \tau^o(s) p(s)\, ds} \\
&\quad + (1 - F(r_1))(1 - F(r_2)) F(r_3) e^{-\int_{r_3}^{r_o} \tau^o(s) p(s)\, ds} \\
&= F(r_1) \frac{F(r_o)}{F(r_1)} + (1 - F(r_1)) F(r_2) \frac{F(r_o)}{F(r_2)} \\
&\quad + (1 - F(r_1))(1 - F(r_2)) F(r_3) \frac{F(r_o)}{F(r_3)} \\
&= F(r_o)[1 + (1 - F(r_1)) + (1 - F(r_1))(1 - F(r_2))] \\
&= F(r_o)[3 - 2F(r_1) - F(r_2) + F(r_1)F(r_2)].
\end{aligned}
$$

The relationship to be checked is that $K((r_o, r_o, r_o), \tau^o) \geq K((r_1, r_2, r_3), \tau^o)$.

$$
\begin{aligned}
& K((r_o, r_o, r_o), \tau^o) - K((r_1, r_2, r_3), \tau^o) \\
= \;& 3F(r_o) - 3F(r_o)^2 + F(r_o)^3 - [F(r_o)[3 - 2F(r_1) - F(r_2) \\
& + F(r_1)F(r_2)]] \\
= \;& 3F(r_o) + F(r_o)[-3F(r_o) - F(r_o)^2] \\
& - [3F(r_o) + F(r_o)[-2F(r_1) - F(r_2) + F(r_1)F(r_2)]] \\
= \;& 3F(r_o) + F(r_o)[(2 - F(r_o))(1 - F(r_o)) - 2]
\end{aligned}
$$

$$-[3F(r_o) + F(r_o)[(2 - F(r_2))(1 - F(r_1)) - 2]]$$

$$= F(r_o)[(2 - F(r_o))(1 - F(r_o)) + 1]$$

$$-[F(r_o)[(2 - F(r_2))(1 - F(r_1)) + 1]] \geq 0,$$

since $F(r_1)$, $F(r_2) \geq F(r_o)$.

Therefore, $K((r_o, r_o, r_o), \tau^o) \geq K((r_1, r_2, r_3), \tau^o)$.

When n=4, $K((r_o, r_o, r_o, r_o), \tau^o)$ and $K((r_1, r_2, r_3, r_4), \tau^o)$ can be expressed in a manner simplifying comparison.

To begin with,

$$K((r_o, r_o, r_o, r_o), \tau^o)$$

$$= 4F(r_o) + F(r_o)$$

$$*[(2 - F(r_o))(1 - F(r_o))(1 - F(r_o)) + (2 - F(r_o)) - 4]$$

$$= F(r_o) * [(2 - F(r_o))(1 - F(r_o))(1 - F(r_o)) + (2 - F(r_o))].$$

And also,

$$K((r_1, r_2, r_3, r_4), \tau^o)$$

$$= 4F(r_o) + F(r_o)$$

$$*[(2 - F(r_3))(1 - F(r_2))(1 - F(r_1)) + (2 - F(r_1)) - 4]$$

$$= F(r_o) * [(2 - F(r_3))(1 - F(r_2))(1 - F(r_1)) + (2 - F(r_1))].$$

It is now clear that corresponding terms can be distinguished in the above two expressions, with the terms of the second expression no greater than the corresponding ones in the first expression.

When n=5, $K((r_o, r_o, r_o, r_o, r_o), \tau^o)$ and $K((r_1, r_2, r_3, r_4, r_5), \tau^o)$ can be expressed as follows:

$$
\begin{aligned}
& K((r_o, r_o, r_o, r_o, r_o), \tau^o) \\
= \; & 5F(r_o) + F(r_o) \\
& \times [(2 - F(r_o))(1 - F(r_o))^3 + (2 - F(r_o))(1 - F(r_o)) - 4] \\
= \; & F(r_o) \times [(2 - F(r_o))(1 - F(r_o))^3 + (2 - F(r_o))(1 - F(r_o)) + 1].
\end{aligned}
$$

And

$$
\begin{aligned}
& K((r_1, r_2, r_3, r_4, r_5), \tau^o) \\
= \; & 5F(r_o) + F(r_o) \\
& \times [(2 - F(r_4))(1 - F(r_3))(1 - F(r_2))(1 - F(r_1)) \\
& + (2 - F(r_2))(1 - F(r_1)) - 4] \\
= \; & F(r_o) \times [(2 - F(r_4))(1 - F(r_3))(1 - F(r_2))(1 - F(r_1)) \\
& + (2 - F(r_2))(1 - F(r_1)) + 1].
\end{aligned}
$$

When n=6, $K((r_o, r_o, r_o, r_o, r_o, r_o), \tau^o)$ and $K((r_1, r_2, r_3, r_4, r_5, r_6), \tau^o)$ can be expressed as follows:

$$
\begin{aligned}
& K((r_o, r_o, r_o, r_o, r_o, r_o), \tau^o) \\
= \; & 6F(r_o) + F(r_o) \\
& \times [(2 - F(r_o))(1 - F(r_o))^4 + (2 - F(r_o))(1 - F(r_o))^2 \\
& - (2 - F(r_o)) - 6] \\
= \; & F(r_o) \times [(2 - F(r_o))(1 - F(r_o))^4 + (2 - F(r_o))(1 - F(r_o))^2 \\
& - (2 - F(r_o))].
\end{aligned}
$$

And

$$
\begin{aligned}
& K((r_1, r_2, r_3, r_4, r_5, r_6), \tau^o) \\
= \; & 6F(r_o) + F(r_o) \\
& \times [(2 - F(r_5))(1 - F(r_4))(1 - F(r_3))(1 - F(r_2))(1 - F(r_1)) \\
& + (2 - F(r_3))(1 - F(r_2))(1 - F(r_1)) + (2 - F(r_1)) - 6] \\
= \; & F(r_o) \times [(2 - F(r_5))(1 - F(r_4))(1 - F(r_3))(1 - F(r_2))(1 - F(r_1)) \\
& + (2 - F(r_3))(1 - F(r_2))(1 - F(r_1)) + (2 - F(r_1))].
\end{aligned}
$$

When n is arbitrary, $K((r_o, r_o, \cdots, r_o), \tau^o)$ and $K((r_1, r_2, \cdots, r_n), \tau^o)$ can be expressed as follows:

$$
\begin{aligned}
& K((r_o, r_o, \cdots, r_o), \tau^o) \\
& = [1 - (1 - F(r_o))^n] e^{-\int_{r_o}^{r_o} \tau^o(s) p(s)\, ds} \\
& = 1 - (1 - F(r_o))^n.
\end{aligned}
$$

And also,

$$
\begin{aligned}
& K((r_1, r_2, \cdots, r_n), \tau^o) \\
= \; & F(r_1) e^{-\int_{r_1}^{r_o} \tau^o(s) p(s)\, ds} + (1 - F(r_1)) F(r_2) e^{-\int_{r_2}^{r_o} \tau^o(s) p(s)\, ds} \\
& + \cdots + (1 - F(r_1))(1 - F(r_2)) \cdots (1 - F(r_{n-1})) \\
& F(r_n) e^{-\int_{r_n}^{r_o} \tau^o(s) p(s)\, ds}.
\end{aligned}
$$

These two expressions do not provide an opportunity for straightforward comparison. However, equivalent expressions, that happen to pertain respectively to n odd and n even, do provide such an opportunity.

Thus, when n is odd, $K((r_o, r_o, \cdots, r_o), \tau^o)$ and $K((r_1, r_2, \cdots, r_n), \tau^o)$ can be expressed as

$$\begin{aligned}
& K((r_o, r_o, \cdots, r_o), \tau^o) \\
= \; & F(r_o)[(2 - F(r_o))(1 - F(r_o))^{n-2} + (2 - F(r_o))(1 - F(r_o))^{n-4} \\
& + \cdots + (2 - F(r_o))(1 - F(r_o))^3 + (2 - F(r_o))(1 - F(r_o)) + 1].
\end{aligned}$$

And also,

$$\begin{aligned}
& K((r_1, r_2, \cdots, r_n), \tau^o) \\
= \; & F(r_o)[(2 - F(r_{n-1}))(1 - F(r_{n-2})) \cdots (1 - F(r_2))(1 - F(r_1)) \\
& +(2 - F(r_{n-3}))(1 - F(r_{n-4})) \cdots (1 - F(r_2))(1 - F(r_1)) + \cdots \\
& +(2 - F(r_4))(1 - F(r_3))(1 - F(r_2))(1 - F(r_1)) + (2 - F(r_2)) \\
& (1 - F(r_1)) + 1],
\end{aligned}$$

and term-by-term comparison is possible, leading to $K((r_o, r_o, \cdots, r_o), \tau^o)$ greater than or equal to $K((r_1, r_2, \cdots, r_n), \tau^o)$ for n odd, since $F(r_o) \leq F(r_i)$.

When n is even, $K((r_o, r_o, \cdots, r_o), \tau^o)$ and $K((r_1, r_2, \cdots, r_n), \tau^o)$ can be expressed as follows:

$$\begin{aligned}
& K((r_o, r_o, \cdots, r_o), \tau^o) \\
= \; & F(r_o)[(2 - F(r_o))(1 - F(r_o))^{n-2} + (2 - F(r_o))(1 - F(r_o))^{n-4} \\
& + \cdots + (2 - F(r_o))(1 - F(r_o))^4 + (2 - F(r_o))(1 - F(r_o))^2 \\
& +(2 - F(r_o))].
\end{aligned}$$

And also,

$$K((r_1, r_2, \cdots, r_n), \tau^o)$$

$$
\begin{aligned}
= \; & F(r_o)[(2 - F(r_{n-1}))(1 - F(r_{n-2})) \cdots (1 - F(r_2))(1 - F(r_1)) \\
& + (2 - F(r_{n-3}))(1 - F(r_{n-4})) \cdots (1 - F(r_2))(1 - F(r_1)) + \cdots \\
& + (2 - F(r_5))(1 - F(r_4))(1 - F(r_3))(1 - F(r_2))(1 - F(r_1)) \\
& + (2 - F(r_3))(1 - F(r_2))(1 - F(r_1)) + (2 - F(r_1))].
\end{aligned}
$$

Again, $K((r_o, r_o, \cdots, r_o), \tau^o)$ is seen to be no less than $K((r_1, r_2, \cdots, r_n), \tau^o)$ for n even, since $F(r_o) \leq F(r_i)$.

Therefore, $K((r_o, r_o, \cdots, r_o), \tau^o)$ is greater than, equal to $K((r_1, r_2, \cdots, r_n), \tau^o)$ for all cases, so that $(r_o, r_o, \cdots, r_o)$ and τ^o are the saddle point coordinate strategies for the fighter and bomber for the duel in which the fighter has multiple identical missiles.

CHAPTER 7. ALTERNATIVE PAYOFF FUNCTIONS

Introductory Remarks

We have in the preceding chapters obtained saddle point coordinate strategies for a bomber-perspective payoff. We shall now introduce a dual with, an alternative, fighter-perspective, payoff function, and also a duel with a payoff function which combines the viewpoints of the bomber and fighter. We will derive saddle point coordinate strategies for the fighter and bomber for these alternative payoff functions. The previous objective function $M(\sigma, \tau)$ is the probability that the bomber is killed. However, in this chapter, the objective function is changed first to the payoff function $M_I(\sigma, \tau)$, which is the probability that the fighter survives, and then is changed to $M_\theta(\sigma, \tau)$ equal to $(1-\theta)M(\sigma, \tau) + \theta M_I(\sigma, \tau)$, $0 \le \theta \le 1$. We assume that the fighter is not vulnerable after releasing a missile. In the case of the duel with payoff function $M_I(\sigma, \tau)$, the duel is assumed to start at the range r_0 defined in terms of $F(r)$ and $p(r)$ as in the previous chapters. In the case of the duel with payoff function $M_\theta(\sigma, \tau)$, the duel is assumed to start at some range R.

Fighter-Perspective Payoff

We assume that an initial bomber firing burst is not allowed in this section. Let us define $M_I(\sigma, \tau)$ as the probability that the fighter survives, the duel when the strategies σ and τ are used by the two opponents.

Then

$$M_I(\sigma, \tau) = \int_c^{r_o} \epsilon^{-\int_r^{r_o} \tau(s)p(s)\, ds}\, d\sigma(r).$$

A saddle point coordinate strategy for the fighter is

$$\sigma^o(r) : \begin{cases} 0 & \text{for } c \leq r < r_o, \\ 1 & \text{for } r = r_o. \end{cases}$$

A saddle point coordinate strategy for the bomber is

$$\tau^o(r) : \begin{cases} \dfrac{-F'(r)}{F(r)p(r)} & \text{for } c \leq r < r_o, \\ 0 & \text{for } r = r_o. \end{cases}$$

We verify that $M_I(\sigma^o, \tau) = M_I(\sigma^o, \tau^o)$ as follows:

$$\begin{aligned} & M_I(\sigma^o, \tau) - M_I(\sigma^o, \tau^o) \\ &= \int_c^{r_o} \epsilon^{-\int_r^{r_o} \tau(s)p(s)\, ds}\, d\sigma^o(r) - \int_c^{r_o} \epsilon^{-\int_r^{r_o} \tau^o(s)p(s)\, ds}\, d\sigma^o(r) \\ &= \int_c^{r_o} \epsilon^{-\int_{r_o}^{r_o} \tau(s)p(s)\, ds}\, d\sigma^o(r) - \int_c^{r_o} \epsilon^{-\int_{r_o}^{r_o} \tau^o(s)p(s)\, ds}\, d\sigma^o(r) \\ &= \int_c^{r_o} d\sigma^o(r) - \int_c^{r_o} d\sigma^o(r) = 0. \end{aligned}$$

Notice that $M_I(\sigma^o, \tau^o) = M_I(\sigma^o, \tau)$, which means that when the fighter uses the saddle point coordinate strategy (σ^o) then the payoff of the duel will not be changed no matter which strategy the bomber uses, because the fighter will leave from the area after firing its missile at range r_o.

It remains to show that

$$M_I(\sigma^o, \tau^o) \geq M_I(\sigma, \tau^o).$$

To this end note that

$$\begin{aligned}
&M_I(\sigma^o,\tau^o) - M_I(\sigma,\tau^o)\\
&= \int_c^{r_o} e^{-\int_r^{r_o}\tau^o(s)p(s)\,ds}\,d\sigma^o(r) - \int_c^{r_o} e^{-\int_r^{r_o}\tau^o(s)p(s)\,ds}\,d\sigma(r)\\
&= \int_c^{r_o} e^{-\int_{r_o}^{r_o}\tau^o(s)p(s)\,ds}\,d\sigma^o(r) - \int_c^{r_o} e^{-\int_r^{r_o}\tau^o(s)p(s)\,ds}\,d\sigma(r)\\
&= \int_c^{r_o} d\sigma^o(r) - \int_c^{r_o}\frac{F(r_o)}{F(r)}\,d\sigma(r)\\
&= 1 - \int_c^{r_o}\frac{F(r_o)}{F(r)}\,d\sigma(r) \geq 0.
\end{aligned}$$

since $\frac{F(r_o)}{F(r)} \leq 1$, and $\int_c^{r_o}\frac{F(r_o)}{F(r)}\,d\sigma(r) \leq \int_c^{r_o} d\sigma(r) = 1$.

Therefore, $M_I(\sigma^o,\tau^o) \geq M_I(\sigma,\tau^o)$, and σ^o and τ^o are the saddle point coordinate strategies for the fighter and bomber for the fighter-perspective payoff.

Hybrid Payoff

In this section we develop saddle point coordinate strategies for the hybrid payoff function $M_\theta(\sigma,\tau)$. We assume that the duel starts at some range R which is specified below. Now

$$\begin{aligned}
M_\theta(\sigma,\tau) &= \theta M_I(\sigma,\tau) + (1-\theta)M(\sigma,\tau)\\
&= \theta\int_c^{R} e^{-\int_r^{R}\tau(s)p(s)\,ds}\,d\sigma(r)\\
&\quad +(1-\theta)\int_c^{R} F(r)e^{-\int_r^{R}\tau(s)p(s)\,ds}\,d\sigma(r)\\
&= \int_c^{R}[F^\theta(r)]e^{-\int_r^{R}\tau(s)p(s)\,ds}\,d\sigma(r),
\end{aligned}$$

where $F^\theta(r) = \theta + ((1-\theta)F(r))$.

The range $r_o(\theta)$, assumed to be finite for all $\theta < 1$, is given by

$$\int_c^{r_o(\theta)} \frac{-F^{\theta\prime}(r)}{F^{\theta}(r)p(r)}\,dr = A.$$

Now we check that $r_o(\theta)$ increases with θ between 0 and 1, from $r_o(\theta) = r_o$ at $\theta = 0$ to $r_o(\theta) = +\infty$ at $\theta = 1$.

We have

$$\begin{aligned}
&\frac{-F'(r)}{F(r)p(r)} - \left(\frac{-F^{\theta\prime}(r)}{F^{\theta}(r)p(r)}\right) \\
&= \left(\frac{1}{p(r)}\right)\left[\frac{-F'(r)}{F(r)} + \frac{F^{\theta\prime}(r)}{F^{\theta}(r)}\right] \\
&= \left(\frac{1}{p(r)}\right)\left[\frac{-F'(r)}{F(r)} + \frac{(1-\theta)F'(r)}{\theta + (1-\theta)F(r)}\right] \\
&= \left(\frac{1}{p(r)}\right)\left[\frac{-F'(r)}{F(r)} + \frac{F'(r)}{\left(\frac{\theta}{1-\theta}\right) + F(r)}\right] \geq 0,
\end{aligned}$$

since $F'(r) \leq 0$, and $\left(\frac{\theta}{1-\theta}\right) > 0$ for $0 \leq \theta < 1$.

Hence, the range $r_o(\theta)$ is increasing in θ, since $r_o(\theta)$ is defined by

$$\int_c^{r_o(\theta)} \frac{-(1-\theta)F'(r)}{[\theta + (1-\theta)F(r)]p(r)}\,dr = A.$$

Also, since in the limit, as $\theta \to 1$,

$$\frac{-(1-\theta)F'(r)}{\theta + (1-\theta)F(r)} \to 0,$$

the range $r_o(\theta)$ increases up to infinity as θ increases from 0 to 1.

When the duel starts at range $R \geq r_o(\theta)$ for $0 \leq \theta < 1$, saddle point coordinate strategies are given as follows:

A saddle point coordinate strategy for the fighter is

$$\sigma^{\theta}(r) : \begin{cases} \dfrac{p(r_o(\theta))}{p(r)} & \text{for } c \leq r < r_o(\theta), \\ 1 & \text{for } r = r_o(\theta). \end{cases}$$

A saddle point coordinate strategy for the bomber is

$$\tau^{\theta}(r) : \begin{cases} \dfrac{-F^{\theta\prime}(r)}{F^{\theta}(r)p(r)} & \text{for } c \le r < r_O(\theta), \\ 0 & \text{for } r = r_O(\theta). \end{cases}$$

This is verified in exactly the same way as in the section on verification of candidate bomber strategy in Chapter 3.

When the duel starts at range $R < r_O(\theta)$ for $0 \le \theta < 1$, and an initial bomber firing burst is allowed, the payoff function for this model is as follows:

$$M_{\theta}(\sigma, \tau) = \int_c^R F^{\theta}(r) e^{-\int_r^R \tau(s)p(s)\,ds - p(R)a}\, d\sigma(r),$$

where $a = \int_R^{r_O(\theta)} \tau(r)\,dr$.

Then saddle point coordinate strategies of the duel are given as follows:

A saddle point coordinate strategy for the fighter is

$$\sigma^{\theta}(r) : \begin{cases} \dfrac{p(r_O(\theta))}{p(r)} & \text{for } c \le r < R, \\ 1 & \text{for } r = R. \end{cases}$$

A saddle point coordinate strategy for the bomber is

$$\tau^{\theta}(r) : \begin{cases} \dfrac{-F^{\theta\prime}(r)}{F^{\theta}(r)p(r)} & \text{for } c \le r < R, \\ a_O & \text{for } r = R. \end{cases}$$

This is also verified in the same way as in the section on late duel start with initial burst possibility for the bomber in Chapter 3.

As θ increases from 0 to 1, the range $r_O(\theta)$ increases, and the initial bomber firing burst is bigger and the last firing probability $(\frac{p(r_O(\theta))}{p(c)})$ is smaller and the initial firing probability $(1 - (\frac{p(r_O(\theta))}{p(R-)}))$ is larger for the fighter.

When $\theta = 1$, and the duel starts at range $R < r_o(1) = +\infty$, and an initial bomber firing burst is allowed, $M_\theta(\sigma, \tau)$ is changed to $M_{Ib}(\sigma, \tau)$, which is

$$M_{Ib}(\sigma, \tau) = \int_c^R e^{-\int_r^R \tau(s)p(s)\,ds - p(R)a}\, d\sigma(r),$$

where a is defined by

$$a = \int_R^{+\infty} \tau(r)\,dr.$$

Then saddle point coordinate strategies of the duel are given as follows:

A saddle point coordinate strategy for the fighter is

$$\sigma^1(r) : \begin{cases} 0 & \text{for } c \le r < R, \\ 1 & \text{for } r = R. \end{cases}$$

A saddle point coordinate strategy for the bomber is

$$\tau^1(r) : \begin{cases} 0 & \text{for } c \le r < R, \\ A & \text{for } r = R. \end{cases}$$

We verify that $M_{Ib}(\sigma^1, \tau) \ge M_{Ib}(\sigma^1, \tau^1)$.

$$\begin{aligned} & M_{Ib}(\sigma^1, \tau) - M_{Ib}(\sigma^1, \tau^1) \\ = & \int_c^R e^{-\int_R^R \tau(s)p(s)\,ds - p(R)a}\, d\sigma^1(r) \\ & - \int_c^R e^{-\int_R^R 0p(s)\,ds - p(R)A}\, d\sigma^1(r) \\ = & \ e^{-p(R)a} - e^{-p(R)A} \\ = & -(p(R)a - p(R)A)e^{-p(R)A} \\ = & \ (A - a)p(R)e^{-p(R)A} \ge 0, \end{aligned}$$

since $a = \int_R^{+\infty} \tau(r)\,dr \le A$.

We now show that $M_{Ib}(\sigma^1,\tau^1) \geq M_{Ib}(\sigma,\tau^1)$.

$$\begin{aligned} M_{Ib}(\sigma^1,\tau^1) &= \int_c^R e^{-\int_R^R 0p(s)\,ds - p(R)A}\, d\sigma^1(r) \\ &= \int_c^R e^{-p(R)A}\, d\sigma^1(r) \\ &= e^{-p(R)A}, \end{aligned}$$

since $\int_c^R d\sigma^1(r) = 1$.

And also,

$$\begin{aligned} M_{Ib}(\sigma,\tau^1) &= \int_c^R e^{-\int_r^R 0p(s)\,ds - p(R)A}\, d\sigma(r) \\ &= \int_c^R e^{-p(R)A}\, d\sigma(r) \\ &= e^{-p(R)A} \int_c^R d\sigma(r) \\ &= e^{-p(R)A}\sigma(R) \leq e^{-p(R)A} = M_{Ib}(\sigma^1,\tau^1) \end{aligned}$$

Hence, $M_{Ib}(\sigma^1,\tau^1) \geq M_{Ib}(\sigma,\tau^1)$, and σ^1 and τ^1 are the saddle point coordinate strategies for the fighter and bomber for the hybrid payoff function.

CHAPTER 8. NON-ZERO SUM VERSION

Introductory Remarks

This chapter deals with the equilibrium strategies for the non-zero sum version of the classical fighter-bomber duel. Suppose that the duel is a non-zero sum game under the same conditions as those of the fighter-perspective payoff problem, which means that it is no longer true that the payoff to the fighter is equal to the negative value of the payoff to the bomber for all outcomes.

It is also true that some of the results for zero-sum games no longer hold, namely:

(a) a maximin point is not necessarily an equilibrium pair or vice versa,

(b) all equilibrium pairs do not have the same payoffs, and

(c) there is no obvious solution concept for the game [43].

Here, an equilibrium point is defined in the sense of Nash [34]: That is, (σ^*, τ^*) is an equilibrium point if the simultaneous choice is made, neither player will have any cause for changing his mind; that is,

$$M_I(\sigma^*, \tau^*) \geq M_I(\sigma, \tau^*), \ \forall \sigma,$$

$$M_{II}(\sigma^*, \tau^*) \geq M_{II}(\sigma^*, \tau), \ \forall \tau.$$

Also a maximin point is defined to be a strategy pair (σ^*, τ^*) such that σ^* is a maximin strategy for the fighter and τ^* is a maximin strategy for the bomber; that is,

$$\min_{\tau} M_I(\sigma^*, \tau) = \max_{\sigma} \min_{\tau} M_I(\sigma, \tau),$$

and

$$\min_{\sigma} M_{II}(\sigma, \tau^*) = \max_{\tau} \min_{\sigma} M_{II}(\sigma, \tau).$$

However, with respect to (c), if, in this non-zero sum case, we are able to find a strategy pair (σ^*, τ^*) that is both an equilibrium point and a maximin point, then we are able to recommend, and/or anticipate the use of, this pair with almost as much confidence as we can recommend saddle point coordinate strategies in the zero-sum case.

Let us assume that the duel starts at range r_O for this chapter. The objective functions for the players are as follows:

$$\begin{aligned} M_I(\sigma, \tau) &= \int_c^{r_O} e^{-\int_r^{r_O} \tau(s)p(s)\,ds}\, d\sigma(r). \\ M_{II}(\sigma, \tau) &= 1 - \int_c^{r_O} F(r) e^{-\int_r^{r_O} \tau(s)p(s)\,ds}\, d\sigma(r). \end{aligned}$$

The fighter wishes to maximize the probability $M_I(\sigma, \tau)$ of the fighter's own survival. And, similarly, the bomber wants to maximize the probability $M_{II}(\sigma, \tau)$ of the bomber's own survival. We now determine the equilibrium point strategies for both players.

Equilibrium Point Analysis

With $M_I(\sigma, \tau)$ and $M_{II}(\sigma, \tau)$ defined as above, define σ^* and τ^* as:

$$\sigma^*(r) : \begin{cases} 0 & \text{for } c \le r < r_O, \\ 1 & \text{for } r = r_O, \end{cases}$$

$$\tau^*(r) : \begin{cases} \dfrac{-F'(r)}{F(r)p(r)} & \text{for } c \leq r < r_O, \\ 0 & \text{for } r = r_O. \end{cases}$$

Then the pair of strategies (σ^*, τ^*) form an equilibrium point for the bi-matrix game; this is verified as follows:

We verify that $M_I(\sigma^*, \tau^*) \geq M_I(\sigma, \tau^*)$.

$$\begin{aligned} M_I(\sigma^*, \tau^*) &= \int_c^{r_O} e^{-\int_r^{r_O} \tau^*(s)p(s)\,ds}\, d\sigma^*(r) \\ &= e^{-\int_{r_O}^{r_O} \tau^*(s)p(s)\,ds} \\ &= 1. \end{aligned}$$

And

$$\begin{aligned} M_I(\sigma, \tau^*) &= \int_c^{r_O} e^{-\int_r^{r_O} \tau^*(s)p(s)\,ds}\, d\sigma(r) \\ &= \int_c^{r_O} e^{-\int_r^{r_O} \tau^*(s)p(s)\,ds} \\ &= \int_c^{r_O} \frac{F(r_O)}{F(r)}\, d\sigma(r) \leq 1. \end{aligned}$$

Therefore, $M_I(\sigma^*, \tau^*) \geq M_I(\sigma, \tau^*)$, since $\dfrac{F(r_O)}{F(r)} \leq 1$ for $r \leq r_O$, and also $\int_c^{r_O} \frac{F(r_O)}{F(r)}\, d\sigma(r) \leq \int_c^{r_O} d\sigma(r) = 1$.

Now we show that $M_{II}(\sigma^*, \tau^*) \geq M_{II}(\sigma^*, \tau)$.

$$\begin{aligned} M_{II}(\sigma^*, \tau^*) &= 1 - \int_c^{r_O} F(r) e^{-\int_r^{r_O} \tau^*(s)p(s)\,ds}\, d\sigma^*(r) \\ &= 1 - F(r_O) e^{-\int_{r_O}^{r_O} \tau^*(s)p(s)\,ds} \\ &= 1 - F(r_O). \end{aligned}$$

And also,

$$M_{II}(\sigma^*, \tau) = 1 - \int_c^{r_O} F(r) e^{-\int_r^{r_O} \tau(s)p(s)\,ds}\, d\sigma^*(r)$$

$$= 1 - F(r_O)\epsilon^{-\int_{r_O}^{r_O} \tau(s)p(s)\,ds}$$
$$= 1 - F(r_O).$$

Therefore, $M_{II}(\sigma^*, \tau^*) \geq M_{II}(\sigma^*, \tau)$, and the pair of strategies (σ^*, τ^*) is an equilibrium point for the non-zero sum game.

Note that σ^* and τ^* are saddle point coordinate strategies for the duel with objective function equal to the bomber' survival: namely

$$M_{II}(\sigma^*, \tau^*) \geq M_{II}(\sigma^*, \tau).$$

as shown above, and also

$$M_{II}(\sigma^*, \tau^*) \leq M_{II}(\sigma, \tau^*),$$

which is shown as follows:

$$\begin{aligned}
& M_{II}(\sigma^*, \tau^*) - M_{II}(\sigma, \tau^*) \\
= \; & 1 - F(r_O) - [1 - \int_C^{r_O} F(r)\epsilon^{-\int_r^{r_O} \tau^*(s)p(s)\,ds}\, d\sigma(r)] \\
= \; & 1 - F(r_O) - [1 - \int_C^{r_O} F(r)\frac{F(r_O)}{F(r)}\, d\sigma(r)] \\
= \; & 1 - F(r_O) - [1 - F(r_O)\int_C^{r_O} d\sigma(r)] \\
= \; & -F(r_O) + F(r_O)\int_C^{r_O} d\sigma(r) = 0,
\end{aligned}$$

since $\int_C^{r_O} d\sigma(r) = 1$.

Since a saddle point coordinate strategy for a maximizing player is also a maximin strategy, this shows that τ^* is a maximin strategy for the payoff function $M_{II}(\sigma, \tau)$:

$$\min_{\sigma} M_{II}(\sigma, \tau^*) = \max_{\tau} \min_{\sigma} M_{II}(\sigma, \tau).$$

It is equally true that σ^* is a maximin strategy for the payoff function $M_I(\sigma, \tau)$:

$$\min_{\tau} M_I(\sigma^*, \tau) = \max_{\sigma} \min_{\tau} M_I(\sigma, \tau).$$

This follows from the fact, established in the section on fighter-perspective payoff in Chapter 7, that (σ^*, τ^*) is a saddle point of the game with payoff function $M_I(\sigma, \tau)$. We then, as in the case of τ^*, use the fact that a saddle point coordinate strategy for a maximizing player is also a maximin strategy.

Therefore, all told, the equilibrium pair of strategies (σ^*, τ^*) is also a maximin pair of the strategies for the bi-matrix game.

CHAPTER 9. COOPERATIVE VERSION

Introductory Remarks

We consider a sales competition between two businessmen, whom we shall call Mr. Little (fighter) and Mr. Big (bomber), essentially in Karlin's sales competition described in Chapter 2, except that we do not include Karlin's upper bound on Mr. Big's sales efforts. The new feature is that cooperation between the two players is now allowed, and that binding contracts can be made.

We assume that the strategies available to the two players are the strategies σ and τ of Chapter 8, and that no outcome pairs other than the pairs $(M_I(\sigma,\tau), M_{II}(\sigma,\tau))$ of Chapter 8 are possible. The feature that we introduce is that pairs (σ,τ) can be chosen in collaboration, thus eliminating the non-cooperative feature of Chapter 8.

In keeping with Nash [35], we adopt as a solution any pair (σ^*, τ^*) that maximizes the hyperbolic function

$$(M_I(\sigma,\tau) - \max_{\sigma}\min_{\tau} M_I(\sigma,\tau))(M_{II}(\sigma,\tau) - \max_{\tau}\min_{\sigma} M_{II}(\sigma,\tau)),$$

with both factors non-negative. As elaborated on in Akbar [1], this solution concept amounts to postulating that, of two possible points $(M_I(\sigma,\tau), M_{II}(\sigma,\tau))$ in the feasible set S, that point will be preferred by the two parties that is preferred by the player with the higher relative stake in the outcome. We note that, as is typical of cooperative solution concepts, the emphasis now is not so much on what the parties will do, but rather on what the parties will obtain.

The objective functions for each player, as in the previous chapter, are as follows:

$$
\begin{aligned}
M_I(\sigma,\tau) &= \int_c^{r_O} e^{-\int_r^{r_O} \tau(s)p(s)\,ds}\, d\sigma(r). \\
M_{II}(\sigma,\tau) &= 1 - \int_c^{r_O} F(r)e^{-\int_r^{r_O} \tau(s)p(s)\,ds}\, d\sigma(r).
\end{aligned}
$$

Bargaining Solution

We assume that the sales competition begins at the "range" r_O, and also, as in the previous chapter, that the customer susceptibility functions are non-decreasing in time, which, as in Chapter 8, means that $F(r)$ and $p(r)$ are non-increasing as the range increases.

We now show that the guarantee point

$$(X^*, Y^*) \equiv (\max_\sigma \min_\tau M_I(\sigma,\tau), \max_\tau \min_\sigma M_{II}(\sigma,\tau))$$

is the unique bargaining solution.

This we shall do by showing 1. $(X^*,Y^*) \in S$, 2. $X \le X^* = 1$, and 3. $Y \le Y^*$ when $X = X^*$.

1. To begin with, we show that $(X^*,Y^*) \in S$.

Let σ^* and τ^* be as defined in Chapters 7 and 8. It was shown in Chapter 7 that (σ^*,τ^*) is a saddle point of the game, with payoff function $M_I(\sigma,\tau)$ from Mr. Little's perspective, which implies that

$$X^* \equiv \max_\sigma \min_\tau M_I(\sigma,\tau) = M_I(\sigma^*,\tau^*)$$

Again, it was shown in Chapter 8 that (σ^*, τ^*) also is a saddle point of the game, with payoff function $M_{II}(\sigma, \tau)$ from Mr. Big's perspective, and this implies that

$$Y^* \equiv \max_{\tau} \min_{\sigma} M_{II}(\sigma, \tau) = M_{II}(\sigma^*, \tau^*).$$

All told, then,

$$(X^*, Y^*) = (M_I(\sigma^*, \tau^*), M_{II}(\sigma^*, \tau^*)),$$

which implies that $(X^*, Y^*) \in S$.

2. We show next that, for all point (X, Y) in the feasible set S, $X \leq X^* = 1$. We show this by showing that

$$X \equiv M_I(\sigma, \tau) \leq M_I(\sigma^*, \tau^*) \equiv X^* = 1.$$

To begin with,

$$\begin{aligned} X^* &= \max_{\sigma} \min_{\tau} M_I(\sigma, \tau) \\ &= \max_{\sigma} \min_{\tau} \int_c^{r_O} e^{-\int_r^{r_O} \tau(s)p(s)\,ds}\, d\sigma(r) \\ &= \max_{\sigma} \int_c^{r_O} e^{-\int_r^{r_O} \tau^*(s)p(s)\,ds}\, d\sigma(r) \\ &= e^{-\int_{r=r_O}^{r_O} \tau^*(s)p(s)\,ds} \\ &= 1. \end{aligned}$$

And

$$\begin{aligned} X &= M_I(\sigma, \tau) \\ &= \int_c^{r_O} e^{-\int_r^{r_O} \tau(s)p(s)\,ds}\, d\sigma(r) \leq 1, \end{aligned}$$

since $e^{-\int_r^{r_O} \tau(s)p(s)\,ds} \leq 1$.

3. Finally we show that $Y \leq Y^*$ when $X = X^*$.

Consider any (X, Y) with $X = X^* = 1$. Then any (σ, τ) generating that (X, Y) must be such that

$$\int_c^{r_o} e^{-\int_r^{r_o} \tau(s)p(s)\,ds}\, d\sigma(r) = 1,$$

which implies that that (σ, τ) is such that σ assigns unit weight to

$$\{r : \int_r^{r_o} \tau(s)p(s)\,ds = 0\}.$$

Then,

$$\begin{aligned} 1 - Y &\equiv \int_c^{r_o} F(r) e^{-\int_r^{r_o} \tau(s)p(s)\,ds}\, d\sigma(r) \\ &= \int_c^{r_o} F(r)\, d\sigma(r) \geq F(r_o). \end{aligned}$$

Hence, when $X = X^* = 1$,

$$Y \leq 1 - F(r_o) = Y^*.$$

That $X \leq X^*$ and $Y \leq Y^*$ for $X = X^*$ shows that there are no points of S in the close quadrant $(X \geq X^*, Y \geq Y^*)$, other than (X^*, Y^*) itself, so that the maximum of the hyperbolic function over the feasible set S is the value zero achieved by (X^*, Y^*). That the guarantee point (X^*, Y^*) itself is the unique bargaining solution underlines the essential competitiveness of the game in question, which prevents the opponents' improving of their prospects through collaboration, beyond what they can secure for themselves without cooperation.

CHAPTER 10. CONCLUSION

Summary

This research analyzed saddle point coordinate strategies for the fighter–bomber duel in several different situations. At first, the Weiss-Gillman and Karlin models of the classical fighter-bomber duel are examined, with respect to the possible extension of the class of saddle point coordinate strategies for the fighter. It is noted that the solutions for the Weiss-Gillman can be obtained when the duel starts before a certain natural range r_O and after r_O, and that a solution for the Karlin models can be obtained when the duel starts before r_O if an altered restriction is met.

A late duel start model is discussed with special structure, for which an initial bomber firing burst is not allowed.

A saddle point coordinate strategy for the extension of the duel in which the fighter has multiple identical missiles is found.

Saddle point coordinate strategies for the fighter and bomber are discussed for a fighter-perspective payoff. Saddle point coordinate strategies are found for a certain hybrid payoff function consisting of a linear combination of bomber– and fighter–perspective payoff functions. The behaviorally plausible pattern emerges that, as the value of the fighter relative to the bomber is made to increase, discrete probability mass in the fighter's optimal strategy shifts from latest–possible firing to earliest–

possible firing.

For the non-zero sum version of the fighter-bomber duel, a pair of strategies $(\sigma^{\times}, \tau^{\times})$ is found to be both an equilibrium point and a maximin point, making $(\sigma^{\times}, \tau^{\times})$ an especially plausible solution construct.

A cooperative version of the duel (in commercial competition form) is studied. It is found that the duel, as originally perceived with no cooperation in mind, is inherently so competitive that the players can be expected to gain nothing from cooperation, and to fall back on payoffs that they can guarantee for and by themselves.

Recommendations for Further Study

In Chapter 3, under the assumption that initial burst possibility for the bomber is allowed, a saddle point coordinate strategy for the fighter and bomber is found. Saddle points of the duel also should be studied for the case where a burst is possible at any time.

In Chapter 6, the saddle point coordinate strategies for the fighter and bomber are applied to the duel in which the fighter has several identical missiles. A possible extension of this research is to study the case of several non–identical missiles.

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